

FINITE LOCALITIES III

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Introduction

This paper is the third in a series on finite localities, whose earlier installments are Part I [Ch2] and Part II [Ch3]. We continue the convention of referring to results in earlier parts by prefixing a “I” or (now) a “II” to citations. For example, “II.2.4” refers to the definition 2.4 of proper locality in [Ch3]. Familiarity with the earlier parts is assumed, but section 1 will provide a brief review of some of the core material.

This Part III relies more heavily on the language of fusion systems than do Parts I and II. In particular, if \mathcal{F} is the fusion system of a proper locality (\mathcal{L}, Δ, S) , then the set \mathcal{F}^q of \mathcal{F} -quasicentric subgroups of S and, more importantly, the set \mathcal{F}^s of \mathcal{F} -subcentric subgroups of S will be seen here to play a key role in understanding the structure of \mathcal{L} . These collections, along with the set \mathcal{F}^c of \mathcal{F} -centric subgroups, and the set \mathcal{F}^{cr} of \mathcal{F} -centric radical subgroups of S were defined in II.1.8.

Throughout most of this paper (until the notion of semi-regular locality is introduced at the very end) (\mathcal{L}, Δ, S) will be a proper locality on \mathcal{F} . Thus, (\mathcal{L}, Δ, S) is a locality, $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ is its fusion system, and it is assumed that the set \mathcal{F}^{cr} of \mathcal{F} -centric radical subgroups of S is contained in Δ , and that the normalizer subgroups $N_{\mathcal{L}}(P)$ for $P \in \Delta$ are of characteristic p (whence $\Delta \subseteq \mathcal{F}^s$, by II.2.8). Theorems II.A1 and II.A2 showed how one may alter the set Δ to “restrict” or “expand” \mathcal{L} , while preserving \mathcal{F} , and while preserving also the structure of the poset of partial normal subgroups of \mathcal{L} . These two theorems (or more briefly, their conjunction into a single Theorem A) provide the flexibility whereby Δ may be chosen to be whatever \mathcal{F} -closed collection of subgroups of S is most convenient for a particular analysis. Theorem II.B provided information concerning the special properties possessed by the fusion system \mathcal{F} of a proper locality.

The principal aim of this paper is to show that there is an \mathcal{F} -closed collection of subgroups of S , to be labelled $\delta(\mathcal{F})$, which is preferential for the following two (inter-connected) reasons.

- (1) If $\Delta = \delta(\mathcal{F})$ then each partial normal subgroup $\mathcal{N} \trianglelefteq \mathcal{L}$ is itself a proper locality.

- (2) If $\Delta = \delta(\mathcal{F})$ then, for each partial normal subgroup $\mathcal{N} \trianglelefteq \mathcal{L}$, $C_{\mathcal{L}}(\mathcal{N})$ is a partial normal subgroup of \mathcal{L} , where $C_{\mathcal{L}}(\mathcal{N})$ is the set of elements $g \in \mathcal{L}$ such that x^g is defined and is equal to x for all $x \in \mathcal{N}$.

For any locality (\mathcal{L}, Δ, S) there is an action of S on \mathcal{L} by conjugation, and we may therefore speak of $C_S(X)$ for any non-empty subset X of \mathcal{L} . Let us say that a partial normal subgroup $\mathcal{M} \trianglelefteq \mathcal{L}$ is *large* if $C_S(\mathcal{M}) \leq \mathcal{M}$. One way to define the set $\delta(\mathcal{F})$ is to begin by expanding \mathcal{L} to a proper locality $(\mathcal{L}^s, \mathcal{F}^s, S)$ on \mathcal{F} whose set of objects is as large as possible. That is, we may employ Theorem II.A1 in order to obtain the unique (up to isomorphism) proper locality on \mathcal{F} whose set of objects is the set \mathcal{F}^s of \mathcal{F} -subcentric subgroups of S . Then define $F^*(\mathcal{L}^s)$ to be the intersection of the set of all large partial normal subgroups \mathcal{M} of \mathcal{L}^s containing $O_p(\mathcal{F})$.

$$F^*(\mathcal{L}^s) = \bigcap \{ \mathcal{M} \trianglelefteq \mathcal{L}^s \mid O_p(\mathcal{F})C_S(\mathcal{M}) \leq \mathcal{M} \}.$$

Now define $\delta(\mathcal{F})$ to be the set of all subgroups P of S such that $P \cap F^*(\mathcal{L}^s) \in \mathcal{F}^s$. We show in lemma 6.10 that $\mathcal{F}^{cr} \subseteq \delta(\mathcal{F}) \subseteq \mathcal{F}^s$ and that $\delta(\mathcal{F})$ is \mathcal{F} -closed. Thus $\delta(\mathcal{F})$ is a permissible choice for Δ ; and we say that the proper locality (\mathcal{L}, Δ, S) on \mathcal{F} is *regular* if $\Delta = \delta(\mathcal{F})$. Notice that $\delta(\mathcal{F}) = \mathcal{F}^s$ in the case that \mathcal{L} is simple (i.e. where \mathcal{L} has exactly two partial normal subgroups).

Theorem C. *Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} , and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} . Set $T = S \cap \mathcal{N}$ and let $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$ be the fusion system on T generated by the conjugation maps $c_g : V \rightarrow T$ with $g \in \mathcal{N}$ and with $V \leq T$. Define $C_{\mathcal{L}}(\mathcal{N})$ to be the set of all $h \in \mathcal{L}$ such that, for all $g \in \mathcal{N}$, the product $[g, h] = g^{-1}h^{-1}gh$ is defined and is equal to $\mathbf{1}$. Then the following hold.*

- (1) $\delta(\mathcal{E}) = \{V \leq T \mid C_S(\mathcal{N})V \in \delta(\mathcal{F})\}$, and \mathcal{N} is itself a regular locality on \mathcal{E} .
- (2) $C_{\mathcal{L}}(\mathcal{N}) \trianglelefteq \mathcal{L}$, and $C_{\mathcal{L}}(\mathcal{N}) = O^p(C_{\mathcal{L}}(T))C_S(\mathcal{N})$.

For any proper locality (\mathcal{L}, Δ, S) on \mathcal{F} , define $F^*(\mathcal{L})$ to be the partial normal subgroup $\mathcal{L} \cap F^*(\mathcal{L}^s)$ of \mathcal{L} corresponding to the partial normal subgroup $F^*(\mathcal{L}^s)$ of \mathcal{L}^s via Theorem II.A2. A partial subgroup \mathcal{K} of \mathcal{L} is defined to be *subnormal* in \mathcal{L} if there is a sequence $(\mathcal{N}_0, \dots, \mathcal{N}_m)$ of partial subgroups of \mathcal{L} with $\mathcal{K} = \mathcal{N}_0 \trianglelefteq \dots \trianglelefteq \mathcal{N}_m = \mathcal{L}$. The partial subnormal subgroup \mathcal{K} of \mathcal{L} is a *component* of \mathcal{L} if $\mathcal{K} = O^p(\mathcal{K})$ and $\mathcal{K}/Z(\mathcal{K})$ is simple (where $Z(\mathcal{K}) = C_S(\mathcal{K}) \cap \mathcal{K}$).

For subsets X and Y of a partial group, write $[X, Y] = \mathbf{1}$ to indicate that $[x, y]$ is defined and is equal to $\mathbf{1}$ for all $x \in X$ and all $y \in Y$.

Theorem D. *Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} and let $(\mathcal{K}_1, \dots, \mathcal{K}_n)$ be a non-redundant list of all of the components of \mathcal{L} . Let $E(\mathcal{L})$ be the smallest partial subgroup of \mathcal{L} containing all of the components \mathcal{K}_i . Then $E(\mathcal{L}) \trianglelefteq \mathcal{L}$, and*

$$F^*(\mathcal{L}) = O_p(\mathcal{F})E(\mathcal{L}) = O_p(\mathcal{F})\mathcal{K}_1 \cdots \mathcal{K}_n,$$

where $[O_p(\mathcal{F}), E(\mathcal{L})] = \mathbf{1}$, and where $[\mathcal{K}_i, \mathcal{K}_j] = \mathbf{1}$ for all i and j with $i \neq j$. Further, we have $C_S(F^*(\mathcal{L})) \leq F^*(\mathcal{L})$, and $\mathcal{L} = F^*(\mathcal{L})H$, where $H := N_{\mathcal{L}}(S \cap F^*(\mathcal{L}))$ is a subgroup of \mathcal{L} which acts on \mathcal{L} and (by restriction) on $F^*(\mathcal{L})$ by everywhere-defined conjugation.

Theorem E. Let \mathbb{L}_δ be the full sub-category of the category of partial groups, whose class of objects is the class of regular localities. Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} .

- (a) If $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup of \mathcal{L} then the inclusion map $\mathcal{N} \rightarrow \mathcal{L}$ is a \mathbb{L}_δ -homomorphism.
- (b) If $X \leq S$ is fully normalized in \mathcal{F} , and $X \leq F^*(\mathcal{L})$, then there is a unique regular locality \mathcal{L}_X on $N_{\mathcal{F}}(X)$ such that \mathcal{L}_X is a subset of \mathcal{L} , and such that the inclusion map $\mathcal{L}_X \rightarrow \mathcal{L}$ is an \mathbb{L}_δ -homomorphism.
- (c) If $Z \leq C_S(O^p(\mathcal{L}))$ is a normal subgroup of \mathcal{L} then the quotient locality \mathcal{L}/Z is regular, and the quotient map $\mathcal{L} \rightarrow \mathcal{L}/Z$ is an \mathbb{L}_δ -homomorphism.

It is a defect of the category \mathbb{L}_δ that, aside from some special cases (such as point (c) of Theorem E), homomorphic images of regular localities need not be regular - or even proper. One way in which to address this defect (but which will not be pursued here) is as follows. Define a locality $(\mathcal{L}, \Delta, \Sigma)$ to be *semi-regular* if there exists a regular locality $\tilde{\mathcal{L}}$ and a projection (cf. I.4.4) $\rho : \tilde{\mathcal{L}} \rightarrow \mathcal{L}$. Composites of projections are projections, so the category \mathbb{L}_σ of partial groups whose objects are semi-regular localities is closed with respect to projections. Further, the Correspondence Theorem I.4.7 shows that partial normal subgroups of semi-regular localities are themselves semi-regular localities. What is then needed is a version of point (b) in Theorem E, in order to obtain the beginnings of a satisfactory category. But before undertaking such an investigation, we shall turn, in Part IV, to the study of normalizers and centralizers of images of certain \mathbb{L}_δ -homomorphisms.

Section 1: Localizable pairs

We assume that the reader has become comfortable with the basic notions introduced in the first section of Part I; specifically partial groups, partial subgroups, partial normal subgroups, and homomorphisms of partial groups. These notions will not be reviewed here.

From section 2 of Part I: A partial group \mathcal{L} is *objective* if there is a set Δ of subgroups of \mathcal{L} (a set of *objects*) which defines the domain \mathbf{D} of the product Π in \mathcal{L} , and which satisfies a pair of closure conditions as expressed in definition II.2.1. A (finite) *locality* is a finite, objective partial group (\mathcal{L}, Δ) such that Δ is a set of subgroups of some $S \in \Delta$, S is a p -group for some prime p , and S is maximal in the poset of p -subgroups of \mathcal{L} .

A fundamental property of localities is given by I.2.6 and I.2.7. Namely, let (\mathcal{L}, Δ, S) be a locality, and let $w = (g_1, \dots, g_n)$ be a word in the free monoid $\mathbf{W}(\mathcal{L})$. Define S_w to be the set of all elements $x = x_0 \in S$ such that x_0 is conjugated to an element x_1 of S by g_1 , x_1 is conjugated to an element x_2 of S by g_2 , and so on. Then

- (*) S_w is a subgroup of S , and $w \in \mathbf{D}$ if and only if $S_w \in \Delta$.

This result is involved in virtually every argument, and should require (and will receive) no further reference. In connection with (*) one says of a word $w \in \mathbf{W}(\mathcal{L})$ that w is in \mathbf{D} via P if $P \in \Delta$ and $P \leq S_w$. A trivial consequence of (*) is:

- (**) If $w \in \mathbf{D}$ then $S_w \leq S_{\Pi(w)}$.

Part II introduced the relationship between a locality (\mathcal{L}, Δ, S) and its fusion system $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$. Thus, \mathcal{F} is the fusion system on S whose isomorphisms are compositions of restrictions of conjugation maps $c_g : S_g \rightarrow S_{g^{-1}}$ for $g \in \mathcal{L}$.

Most readers will have at least some familiarity with general fusion systems, but sections 1 and 6 of Part II provide all that will be needed here. The notion of “saturation” appears in the proof of Theorem II.6.1, but will play no further role. As mentioned in the introduction, the notion of \mathcal{F} -closed set of subgroups of S (\mathcal{F} a fusion system on S), and the definitions of the sets \mathcal{F}^{cr} , \mathcal{F}^c , \mathcal{F}^q , and \mathcal{F}^s appear early in section II.2. Theorem II.6.2 (due to Henke [He2]) establishes the sequence of inclusions

$$\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q \subseteq \mathcal{F}^s,$$

and establishes that \mathcal{F}^s is \mathcal{F} -closed.

The locality (\mathcal{L}, Δ, S) on \mathcal{F} is defined to be proper (cf. II.2.4) if $\mathcal{F}^{cr} \subseteq \Delta$, and if each of the groups $N_{\mathcal{L}}(P)$ for $P \in \Delta$ is of characteristic p . Lemma II.2.8 provides a “dictionary”, translating back and forth between fusion-theoretic conditions on a given $P \in \Delta$ and properties of $N_{\mathcal{L}}(P)$. Other basic notions from Parts I and II will be recalled as needed.

For any locality (\mathcal{L}, Δ, S) , define

$$O_p(\mathcal{L}) = \bigcap \{S_w \mid w \in \mathbf{W}(\mathcal{L})\}.$$

Equivalently (by I.2.14) $O_p(\mathcal{L})$ is the largest subgroup of S which is a partial normal subgroup of \mathcal{L} . Then $O_p(\mathcal{L}) \leq O_p(\mathcal{F})$, where \mathcal{F} is the fusion system of \mathcal{L} . In the case that \mathcal{L} is proper one has the important equality $O_p(\mathcal{L}) = O_p(\mathcal{F})$ (cf. II.2.3).

Definition 1.1. Let (\mathcal{L}, Δ, S) be a locality and let $\mathcal{H} \leq \mathcal{L}$ be a partial subgroup of \mathcal{L} . Set $T = S \cap \mathcal{H}$, let Γ be a set of subgroups of T , and set

$$\mathcal{H}_{\Gamma} = \{h \in \mathcal{H} \mid S_h \cap T \in \Gamma\}.$$

Let $\mathcal{E} := \mathcal{F}_T(\mathcal{H}, \Gamma)$ be the fusion system on T generated by the set of conjugation maps $c_h : S_h \cap T \rightarrow T$ for $h \in \mathcal{H}_{\Gamma}$. Then (\mathcal{H}, Γ) is a *localizable pair* in \mathcal{L} if:

- (1) Γ is \mathcal{E} -closed.
- (2) T is maximal in the poset of p -subgroups of \mathcal{H} .
- (3) $w \in \mathbf{W}(\mathcal{H})$ and $S_w \cap T \in \Gamma \implies w \in \mathbf{D}(\mathcal{L})$.

Lemma 1.2. Let (\mathcal{L}, Δ, S) be a locality, and let (\mathcal{H}, Γ) be a localizable pair in \mathcal{L} . Set $T = S \cap \mathcal{H}$ and $\mathcal{E} = \mathcal{F}_T(\mathcal{H}, \Gamma)$. Further, set

$$\mathcal{H}_{\Gamma} = \{h \in \mathcal{H} \mid S_h \cap T \in \Gamma\},$$

and

$$\mathbf{D}(\mathcal{H}_{\Gamma}) = \{w \in \mathbf{W}(\mathcal{H}_{\Gamma}) \mid S_w \cap T \in \Gamma\}.$$

Then \mathcal{H}_Γ is a partial group with respect to the restriction of the product $\Pi : \mathbf{D}(\mathcal{L}) \rightarrow \mathcal{L}$ to a mapping $\mathbf{D}(\mathcal{H}_\Gamma) \rightarrow \mathcal{H}_\Gamma$, and with respect to the restriction to \mathcal{H}_Γ of the inversion in \mathcal{L} . Moreover, $(\mathcal{H}_\Gamma, \Gamma, T)$ is then a locality on \mathcal{E} .

Proof. We must verify that the conditions (1) through (4) in the definition (I.1.1) of a partial group are satisfied by \mathcal{H}_Γ and the given product and inversion. Thus \mathcal{H}_Γ is the set of words of length 1 in $\mathbf{D}(\mathcal{H}_\Gamma)$, and it is plain that if the word $u \circ v$ is in $\mathbf{D}(\mathcal{H}_\Gamma)$ then so are u and v . Thus the condition (1) holds. If $w \in \mathbf{D}(\mathcal{H}_\Gamma)$ then $w \in \mathbf{D}$, and $\Pi(w)$ is in \mathcal{H} since \mathcal{H} is a partial subgroup of \mathcal{L} . Then $\Pi(w) \in \mathcal{H}_\Gamma$ since $S_w \leq S_{\Pi(w)}$, and thus Π restricts to a mapping $\mathbf{D}(\mathcal{H}_\Gamma) \rightarrow \mathcal{H}_\Gamma$. The conditions 1.1(2) and (3) are then inherited from Π . For any $h \in \mathcal{H}_\Gamma$ we have $S_{h^{-1}} \cap T = (S_h \cap T)^h$, so \mathcal{H}_Γ is closed under the inversion in \mathcal{L} . The required condition (4), that $\Pi(w^{-1} \circ w) = \mathbf{1}$ for $w \in \mathbf{D}(\mathcal{H}_\Gamma)$, is then immediate. Thus \mathcal{H}_Γ is a partial group.

The inclusion $\mathcal{H}_\Gamma \rightarrow \mathcal{L}$ is a homomorphism since the product in \mathcal{H}_Γ is the restriction of the product Π . Moreover it is for this same reason, and because Γ is \mathcal{E} -closed, that $(\mathcal{H}_\Gamma, \Gamma)$ is an objective partial group. Notice that any subgroup of \mathcal{H}_Γ is also a subgroup of \mathcal{H} . Then 1.1(2) implies that T is maximal in the poset of p -subgroups of \mathcal{H}_Γ , and thus $(\mathcal{H}_\Gamma, \Gamma, T)$ is a locality. By the definition of \mathcal{E} we have $\mathcal{E} = \mathcal{F}_T(\mathcal{H}_\Gamma)$. \square

Remark. We have already, in II.2.11, encountered one sort of localizable pair. Namely, if Γ is an \mathcal{F} -closed subset of Δ then (\mathcal{L}, Γ) is a localizable pair, and $(\mathcal{L}_\Gamma, \Gamma, S)$ is the restriction $\mathcal{L} \upharpoonright_\Gamma$ of \mathcal{L} to Γ .

Lemma 1.3. *Let the hypothesis and notation be as in 1.2, and let $\mathcal{F}_T(\mathcal{H})$ be the fusion system on T generated by the conjugation maps $c_h : S_h \cap T \rightarrow T$ for all $h \in \mathcal{H}$. In order for $(\mathcal{H}_\Gamma, \Gamma, T)$ to be a locality on $\mathcal{F}_T(\mathcal{H})$ it suffices that Γ contain \mathcal{E}^{cr} , and that $\mathcal{F}_T(\mathcal{H})$ be Γ -generated.*

Proof. Set $\mathcal{E} = \mathcal{F}_T(\mathcal{H}_\Gamma)$ and $\mathcal{E}_1 = \mathcal{F}_T(\mathcal{H})$. Then \mathcal{E} is a fusion subsystem of \mathcal{E}_1 . Let \mathcal{E}_2 be the fusion subsystem of \mathcal{E} generated by the set of all \mathcal{E} -automorphisms of members $U \in (\mathcal{E}_1)^{cr}$ such that U is fully normalized in \mathcal{E}_1 and such that $U = O_p(N_{\mathcal{E}_1}(U))$. If $(\mathcal{E}_1)^{cr} \subseteq \Gamma$ then \mathcal{E}_2 is a subsystem of \mathcal{E} , and if \mathcal{E}_1 is Γ -generated then $\mathcal{E}_1 = \mathcal{E}_2$. Thus $\mathcal{E} = \mathcal{E}_1$ if the two stated conditions are fulfilled. \square

Lemma 1.4. *Let (\mathcal{L}, Δ, S) be a proper locality on \mathcal{F} , and let $V \leq S$ be fully normalized in \mathcal{F} . Set*

$$\Gamma = \{X \in \Delta \mid V \trianglelefteq X\}, \quad \text{and} \quad \Sigma = \{Y \leq C_S(V) \mid VY \in \Gamma\}.$$

- (a) *If $\mathcal{F}^c \subseteq \Delta$ then $N_{\mathcal{F}}(V)^{cr} \subseteq \Gamma$ and $C_{\mathcal{F}}(V)^{cr} \subseteq \Sigma$.*
- (b) *If $N_{\mathcal{F}}(V)^{cr} \subseteq \Gamma$ then $(N_{\mathcal{L}}(V), \Gamma)$ is a localizable pair and $(N_{\mathcal{L}}(V)_\Gamma, \Gamma, N_S(V))$ is a proper locality on $N_{\mathcal{F}}(V)$.*
- (c) *If $C_{\mathcal{F}}(V)^{cr} \subseteq \Sigma$ then $(C_{\mathcal{L}}(V), \Sigma)$ is a localizable pair and $(C_{\mathcal{L}}(V)_\Sigma, \Sigma, C_S(V))$ is a proper locality on $C_{\mathcal{F}}(V)$.*

Proof. Set $\mathcal{H} = N_{\mathcal{L}}(V)$ and $\mathcal{K} = C_{\mathcal{L}}(V)$. Then \mathcal{H} is a partial subgroup of \mathcal{L} , and $\mathcal{K} \trianglelefteq \mathcal{H}$. Set $\mathcal{E} = N_{\mathcal{F}}(V)$ and $\mathcal{D} = C_{\mathcal{F}}(V)$. Then \mathcal{E} and \mathcal{D} are (cr) -generated by II.6.1. Then \mathcal{H}_Γ is a locality on \mathcal{E} if $\Gamma \subseteq \Delta$, and \mathcal{K}_Σ is a locality on \mathcal{D} if $X\Sigma \subseteq \Delta$, by 1.3.

It remains to show that the locality \mathcal{H}_Γ is proper if $\mathcal{E}^{cr} \subseteq \Gamma$, and that \mathcal{K}_Σ is proper if $\mathcal{D}^{cr} \subseteq \Sigma$. Under these hypotheses on Γ and Σ , it is only necessary to show that the normalizers of objects in \mathcal{H}_Γ or in \mathcal{K}_Σ are of characteristic p . As $\Gamma \subseteq \Delta$, and since

$$N_{\mathcal{H}}(X) = N_{N_{\mathcal{L}}(X)}(V)$$

for $X \in \Gamma$, it follows from II.2.7(b) that $N_{\mathcal{H}}(X)$ is of characteristic p . For $Y \in \Sigma$ we have $VY \in \Delta$, and so

$$N_{\mathcal{K}}(Y) = C_{N_{\mathcal{H}}(VY)}(V) \trianglelefteq N_{\mathcal{H}}(VY).$$

Then $N_{\mathcal{K}}(Y)$ is of characteristic p by II.2.7(a), and the proof is complete. \square

Corollary 1.5. *Let (\mathcal{L}, Δ, S) be a proper locality on \mathcal{F} , and let $T \leq S$ be strongly closed in \mathcal{F} . Set $\mathcal{L}_T = N_{\mathcal{L}}(T)$ and $\mathcal{C}_T = C_{\mathcal{L}}(T)$.*

- (a) *$(\mathcal{L}_T, \Delta, S)$ is a proper locality on $N_{\mathcal{F}}(T)$.*
- (b) *Set $\Sigma = \{V \leq C_S(T) \mid VT \in \Delta\}$, and assume that $C_{\mathcal{F}}(T)^{cr} \subseteq \Sigma$. Then $(\mathcal{C}_T, \Sigma, C_S(T))$ is a proper locality on $C_{\mathcal{F}}(T)$. Moreover, the condition $C_{\mathcal{F}}(T)^{cr} \subseteq \Sigma$ is fulfilled if $\mathcal{F}^c \subseteq \Delta$.*
- (c) *Let $(\mathcal{L}^+, \Delta^+, S)$ be an expansion of \mathcal{L} and set $\mathcal{L}_T^+ = N_{\mathcal{L}^+}(T)$. Then $(\mathcal{L}_T^+, \Delta^+, S)$ is an expansion of $(\mathcal{L}_T, \Delta, S)$.*

Proof. As T is strongly closed in \mathcal{F} one observes that

$$N_{\mathcal{F}}(T)^{cr} = \{P \in \mathcal{F}^{cr} \mid T \leq P\},$$

and hence $N_{\mathcal{F}}(T)^{cr} \subseteq \Delta$. Then 1.4(a) yields (a). Point (b) is immediate from 1.4(b), and point (c) is immediate from (a). \square

Section 2: The basic setup

Most of the results to be proved in sections 2 through 5 will be concerned with a single partial normal subgroup \mathcal{N} of \mathcal{L} . (The only exceptions are 2.10 and 2.11, which concern pairs of partial normal subgroups.) The following notation will remain fixed.

2.1 (Basic setup). (\mathcal{L}, Δ, S) is a proper locality on \mathcal{F} , and $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup of \mathcal{L} . Set $T = S \cap \mathcal{N}$, and let \mathcal{E} be the fusion system $\mathcal{F}_T(\mathcal{N})$ on T , generated by the conjugation maps $c_g : S_g \cap T \rightarrow T$ for $g \in \mathcal{N}$. Further, set $\mathcal{L}_T = N_{\mathcal{L}}(T)$ and $\mathcal{C}_T = C_{\mathcal{L}}(T)$.

Recall from I.3.1 that T is strongly closed in \mathcal{F} , and T is maximal in the poset of p -subgroups of \mathcal{N} . There is no reason to suppose that \mathcal{E} should be inductive (see II.1.11), or that \mathcal{E} should remain invariant under the process $\mathcal{N} \mapsto \mathcal{N}^+$ of expansion given by Theorem II.A2. Indeed, this non-rigidity of \mathcal{E} relative to \mathcal{F} will be the source of most of the technical difficulties that will be encountered.

Lemma 2.2. *Assume the setup of 2.1, set $\tilde{T} = C_S(T)T$, and set $H = N_{\mathcal{L}}(\tilde{T})$. Then the following hold.*

- (a) *H is a subgroup of \mathcal{L} , and $O_p(H) = O_p(N_{\mathcal{F}}(\tilde{T})) \in \mathcal{F}^{cr}$.*
- (b) *\tilde{T} is strongly closed in \mathcal{F} .*
- (c) *$\mathcal{F}_S(H) = N_{\mathcal{F}}(\tilde{T})$.*
- (d) *$\mathcal{L}_T = \mathcal{C}_T H$.*

Proof. Set $\tilde{T} = C_S(T)T$, and let $\tilde{\mathcal{N}}$ be the partial subgroup $\langle \mathcal{N}, \mathcal{C}_T \rangle$ of \mathcal{L} generated by \mathcal{N} and \mathcal{C}_T . Then $\tilde{\mathcal{N}} \trianglelefteq \mathcal{L}$ and $\tilde{T} = S \cap \tilde{\mathcal{N}}$, by I.5.5. Then (b) follows from I.3.1(a), with $\tilde{\mathcal{N}}$ in the role of \mathcal{N} . Point (d) is immediate from the Frattini Lemma (I.3.11).

Clearly \tilde{T} is \mathcal{F} -centric. Set $Q = O_p(N_{\mathcal{F}}(\tilde{T}))$. Then $Q \trianglelefteq S$, so $\tilde{T} \trianglelefteq O_p(N_{\mathcal{F}}(Q))$, and hence $Q = O_p(N_{\mathcal{F}}(Q))$. Thus

$$(*) \quad Q \in \mathcal{F}^{cr},$$

so $Q \in \Delta$, and $N_{\mathcal{L}}(Q)$ is a subgroup of \mathcal{L} .

By 1.5(a), (H, Δ, S) is a proper locality on $N_{\mathcal{F}}(\tilde{T})$. This yields (c), and II.2.3 shows that $O_p(H) = Q$. Then H is the subgroup $N_{N_{\mathcal{L}}(Q)}(\tilde{T})$ of \mathcal{L} . This result, together with (*), completes the proof of (a). \square

Lemma 2.3. *Assume the setup of 2.1. Let \mathcal{H} be a partial subgroup of \mathcal{L}_T having the property that $(h^{-1}, x, h) \in \mathbf{D}$ for all $x \in \mathcal{N}$ and all $h \in \mathcal{H}$, and let $\lambda : \mathcal{H} \rightarrow \text{Aut}(T)$ be the homomorphism which sends $h \in \mathcal{H}$ to conjugation by h . Then $\text{Im}(\lambda) \subseteq \text{Aut}(\mathcal{E})$.*

Proof. Let $\phi : U \rightarrow U'$ be an \mathcal{E} -isomorphism. By definition, there is a sequence

$$U = U_0 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_n} U_n = U'$$

of \mathcal{E} -isomorphisms, such that each $\phi_i : U_{i-1} \rightarrow U_i$ is the restriction of a conjugation map $c_{x_i} : T \cap S_{g_i} \rightarrow T$, with $x_i \in \mathcal{N}$. Let $h \in \mathcal{H}$. By hypothesis, $(h^{-1}, x_i, h) \in \mathbf{D}$ for all i , and then $(x_i)^h \in \mathcal{N}$ as $\mathcal{N} \trianglelefteq \mathcal{L}$. As $\mathcal{H} \leq N_{\mathcal{L}}(T)$ we may define $V_i := (U_i)^h$, and then define $\psi_i : V_{i-1} \rightarrow V_i$ to be given by conjugation by $(x_i)^h$. Then each ψ_i is an \mathcal{E} -isomorphism, and the composite $\psi := \psi_1 \circ \cdots \circ \psi_n$ is given by $c_{h^{-1}} \circ \phi \circ c_h$ as an \mathcal{E} -isomorphism $U^h \rightarrow (U')^h$. This shows that the conjugation map $c_h : T \rightarrow T$ is \mathcal{E} -fusion-preserving. That is, $c_h \in \text{Aut}(\mathcal{E})$. The verification that the map $h \mapsto c_h$ is a homomorphism $\mathcal{H} \rightarrow \text{Aut}(\mathcal{E})$ is a straightforward application of I.2.3(c). \square

Corollary 2.4. *Suppose that x^h is defined for all $x \in \mathcal{N}$ and all $h \in H := N_{\mathcal{L}}(C_S(T)T)$. Let Γ be a set of subgroups of subgroups of T which is both \mathcal{E} -invariant and $\text{Aut}(\mathcal{E})$ -invariant. Then Γ is \mathcal{F} -invariant.*

Proof. Let $g \in \mathcal{L}$, and let $U \in \Gamma$ with $U \leq S_g \cap T$. Let $\phi : U \rightarrow U^g$ be the conjugation map. Define \mathcal{L}_T and \mathcal{C}_T as in 1.4. By the Splitting Lemma (I.3.12) we may write $g = xf$ where $x \in \mathcal{N}$, $f \in \mathcal{L}_T$, and $S_g = S_{(x,f)}$. As $\mathcal{C}_T \trianglelefteq \mathcal{L}_T$ we may also write $f = yh$ where $y \in \mathcal{C}_T$, $h \in H$, and $S_f = S_{(y,h)}$. Then $S_g = S_{(x,y,h)}$, and ϕ is then the composition of the \mathcal{E} -isomorphism c_x followed by c_h . Then $U^g \in \Gamma$ by 2.3. Then Γ is \mathcal{F} -invariant since \mathcal{F} is generated by the conjugation maps c_g with $g \in \mathcal{L}$. \square

Lemma 2.5. *Assume the setup of 2.1, and let $Q \leq T$ be a subgroup of T such that Q is fully normalized in \mathcal{E} . Suppose that $O_p(\mathcal{L})Q \in \Delta$, and set $K = N_{\mathcal{L}}(Q)$. Then K is a normal subgroup of the group $N_{\mathcal{L}}(Q)$, $N_T(Q)$ is a Sylow subgroup of K , and $N_{\mathcal{E}}(Q) = \mathcal{F}_{N_T(Q)}(K)$.*

Proof. Set $M = N_{\mathcal{L}}(O_p(\mathcal{L})Q)$. As $O_p(\mathcal{L})Q \in \Delta$, M is a subgroup of \mathcal{L} . Then $K = \mathcal{N} \cap N_M(Q) = \mathcal{N} \cap N_{\mathcal{L}}(Q)$ is a normal subgroup of the group $N_{\mathcal{L}}(Q)$ by I.1.8.

Set $X = N_T(V)$ and let Y be a Sylow p -subgroup of K containing X . By I.2.11 there exists $g \in \mathcal{L}$ with $Y^g \leq S$, and then $Y^g \leq T$ since T is strongly closed in \mathcal{F} . Employ the splitting lemma (I.3.12) to obtain $g = fh$ with $f \in \mathcal{N}$, $h \in N_{\mathcal{L}}(T)$, and with $S_g = S_{(f,h)}$. Then $Y^f \leq T$, and so $Y^f \leq N_T(V^f)$. As V is fully normalized in \mathcal{E} , we conclude that $X^f = Y^f$, and thus $X \in \text{Syl}_p(K)$.

Next, let A and B be subgroups of X containing V , such that A and B are conjugate in $N_{\mathcal{E}}(V)$. Let $\gamma : A \rightarrow B$ be an $N_{\mathcal{E}}(V)$ -isomorphism. By the definition of $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$, this means that there exists $w = (f_1, \dots, f_n) \in \mathbf{W}(\mathcal{N})$ such that $A \leq S_w$, and such that γ is given by composing the conjugation maps c_{f_i} . As $O_p(\mathcal{L})V \leq S_w$ we have $w \in \mathbf{D}$, and then $\Pi(w) \in \mathcal{N}$. Then $\Pi(w) \in K$, and thus $N_{\mathcal{E}}(Q) = \mathcal{F}_X(K)$. \square

Recall from II.1.8 that \mathcal{E}^{cr} is defined to be the set of all $U \in \mathcal{E}^c$ such that there exists an \mathcal{E} -conjugate V of U such that V is fully normalized in \mathcal{E} , and such that $V = O_p(N_{\mathcal{E}}(V))$. It has already been remarked that this definition does not quite agree with the usual one (see [AKO], for example). The following lemma provides the justification for this discrepancy. Namely, points (b) and (c) of the lemma establish a “descent” from \mathcal{F}^{cr} to \mathcal{E}^{cr} which relies on our definition, and which would otherwise be lacking.

Lemma 2.6. *Assume the setup of 2.1.*

- (a) *Let $U \leq T$ be a subgroup of T . Assume that U is fully normalized in \mathcal{F} and that $C_T(U) \leq U$. Then $U^{\mathcal{F}} \subseteq \mathcal{E}^c$.*
- (b) *Let $Q \in \mathcal{F}^{cr}$ and set $U = Q \cap T$. Suppose that U is fully normalized in \mathcal{F} . Then $U^{\mathcal{F}} \subseteq \mathcal{E}^c$, U is fully normalized in \mathcal{E} , and $U = O_p(N_{\mathcal{E}}(U))$.*
- (c) *Let $\mathcal{M} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} containing \mathcal{N} . Set $R = S \cap \mathcal{M}$, set $\mathcal{D} = \mathcal{F}_R(\mathcal{M})$, and suppose that \mathcal{D} is inductive. Let $Q \in \mathcal{D}^{cr}$, and set $U = Q \cap T$. Assume that $O_p(\mathcal{L})Q \in \Delta$ and that U is fully normalized in \mathcal{D} . Then $U^{\mathcal{D}} \subseteq \mathcal{E}^c$, U is fully normalized in \mathcal{E} , and $U = O_p(N_{\mathcal{E}}(U))$.*

Proof. Let U be chosen as in (a), and let U' be an \mathcal{F} -conjugate of U . As \mathcal{F} is inductive by II.6.1, there exists an \mathcal{F} -homomorphism $\phi : N_S(U') \rightarrow N_S(U)$ with $U'\phi = U$. As T is strongly closed in \mathcal{F} we obtain $C_T(U')\phi \leq C_T(U)$. As $C_T(U) \leq U$ it follows that $C_T(U') \leq U'$. Since \mathcal{E} -conjugates of \mathcal{F} -conjugates of U are \mathcal{F} -conjugates of U , we obtain $U^{\mathcal{F}} \subseteq \mathcal{E}^c$, and thus (a) holds.

Point (b) is the special case of (c) where $\mathcal{D} = \mathcal{F}$, so it remains only to prove (c). Let \mathcal{M} , R , \mathcal{D} , Q , and U be as stated in (c). Then U is fully normalized in \mathcal{E} by II.1.17. Set $K = N_{\mathcal{M}}(Q)$. By 2.5, with \mathcal{M} in the role of \mathcal{N} , we have $N_R(Q) \in \text{Syl}_p(K)$ and $N_{\mathcal{D}}(Q) = \mathcal{F}_{N_R(Q)}(K)$. Also by 2.5, K is a normal subgroup of the group $N_{\mathcal{L}}(Q)$, where $N_{\mathcal{L}}(Q)$ is a local subgroup of $N_{\mathcal{L}}(O_p(\mathcal{L})Q)$. Then K is of characteristic p by II.2.7, and

hence $O_p(K) = O_p(N_{\mathcal{D}}(Q))$. As $Q \in \mathcal{D}^{cr}$, where \mathcal{D} is inductive, it follows from II.1.14 that $Q = O_p(K)$. Set $D = N_{C_T(U)}(Q)$. Then

$$[Q, D] \leq C_{Q \cap T}(V) = Z(U),$$

and thus D centralizes the chain $Q \geq U \geq 1$ of normal subgroups of K . Then $D \leq Q$ by II.2.7(c), and thus $C_T(U) \leq Q$. Then $C_T(U) \leq Q \cap T = U$. A straightforward variation on the proof of (a) then yields $U^{\mathcal{D}} \subseteq \mathcal{E}^c$.

Set $X = O_p(N_{\mathcal{E}}(U))$. Then $U \leq X \leq T$. By 1.5 the elements of S act as automorphisms of \mathcal{E} by conjugation, so Q acts on $N_{\mathcal{E}}(X)$, and thus X is Q -invariant. By definition, each \mathcal{E} -automorphism of U extends to an \mathcal{E} -automorphism of X , and this translates into the following statement.

- (*) Let β be an \mathcal{E} -automorphism of U , let $\bar{\beta}$ be an extension of β to an \mathcal{E} -automorphism of X , and let $x \in X$. For any $a \in X$, let c_a be the automorphism of U given by conjugation by a . Then $\beta^{-1} \circ c_a \circ \beta = c_{a\bar{\beta}}$. In particular, $\text{Aut}_X(U)$ is a normal subgroup of $\text{Aut}_{\mathcal{E}}(U)$.

Set $K_0 = K \cap \mathcal{N}$ and set $A = N_X(Q)$. Then K_0 is a normal subgroup of K , while (*) yields $\text{Aut}_A(U) \trianglelefteq \text{Aut}_{K_0}(U)$. As $U \in \mathcal{E}^c$, $C_{K_0}(U)$ is the direct product of $Z(U)$ with a normal p' -subgroup of K_0 . Here $O_{p'}(K_0) = 1$ as K is of characteristic p , and so $C_{K_0}(U) = Z(U)$. There is then a natural isomorphism of $\text{Aut}_{K_0}(U)$ with $K_0/Z(U)$, from which it follows that $A/Z(U) \trianglelefteq K_0/Z(U)$. Then $A \trianglelefteq K_0$, so $A \leq O_p(K_0) \leq O_p(K) = Q$. Then $X \leq Q$, and since also $X \leq T$ we arrive at $X \leq U$. Thus $X = U$, completing the proof of (c). \square

For any partial subgroup $\mathcal{H} \leq \mathcal{L}$ define $Z(\mathcal{H})$ to be the set of all $z \in \mathcal{H}$ such that, for all $h \in \mathcal{H}$, h^z is defined and is equal to h .

Lemma 2.7. *Assume the setup of 2.1. Then $Z(\mathcal{N}) = C_T(\mathcal{N})$.*

Proof. As t^z is defined and is equal to t for all $t \in T$ and all $z \in Z(\mathcal{N})$, we have $Z(\mathcal{N}) \leq N_{\mathcal{N}}(T)$. Set $H = N_{\mathcal{L}}(C_S(T)T)$. As \mathcal{L} is proper, I.3.5 yields $N_{\mathcal{N}}(T) \leq H$. As H is a subgroup of \mathcal{L} by 2.2(a), $Z(\mathcal{N})$ is then a normal abelian subgroup of H . As H is of characteristic p , II.2.7 implies that $Z(\mathcal{N})$ is of characteristic p . Thus $Z(\mathcal{N})$ is a p -subgroup of $N_{\mathcal{L}}(T)$. As T is a maximal p -subgroup of \mathcal{N} by I.3.1(a), the lemma follows. \square

For any fusion system \mathcal{F} on a p -group S , and any non-empty set Γ of subgroups of S , define the \mathcal{F} -closure of Γ to be the smallest \mathcal{F} -closed set of subgroups of S containing Γ . Thus, the \mathcal{F} -closure of Γ is the set of subgroups X of S such that X contains an \mathcal{F} -conjugate of a member of Γ .

For any pair Γ and Σ of non-empty sets of subgroups of S , write $\Gamma\Sigma$ for the set of all products XY with $X \in \Gamma$ and $Y \in \Sigma$. If $\Gamma = \{X\}$ is a singleton we may write $X\Sigma$ for $\{X\}\Sigma$.

Lemma 2.8. *Assume the setup of 2.1, and set $X = O_p(\mathcal{L})C_S(\mathcal{N})$. Let Γ be an \mathcal{E} -closed set of subgroups of T containing \mathcal{E}^{cr} , and let Δ_0 be the \mathcal{F} -closure of $X\Gamma$. Assume that $\Delta_0 \subseteq \Delta$ and that \mathcal{E} is Γ -generated. Then (\mathcal{N}, Γ) is a localizable pair, and $(\mathcal{N}_\Gamma, \Gamma, T)$ is a proper locality on \mathcal{E} . Moreover, if $O_p(\mathcal{L})\Gamma \subseteq \Delta$, then the following hold.*

- (a) $\mathcal{F}^{cr} \subseteq \Delta_0 \subseteq \Delta$, the restriction \mathcal{L}_0 of \mathcal{L} to Δ_0 is a proper locality $(\mathcal{L}_0, \Delta_0, S)$ on \mathcal{F} , and $\mathcal{N}_\Gamma = \mathcal{N} \cap \mathcal{L}_0$.
- (b) $N_{\mathcal{L}}(T)$ is a subgroup of \mathcal{L}_0 , and x^g is defined for all $x \in \mathcal{L}_0$ and all $g \in N_{\mathcal{L}}(XT)$. Moreover, the mapping $g \rightarrow c_g$ which sends $g \in N_{\mathcal{L}}(XT)$ to the conjugation map $c_g : \mathcal{L}_0 \rightarrow \mathcal{L}_0$ is a homomorphism $\lambda : N_{\mathcal{L}}(XT) \rightarrow \text{Aut}(\mathcal{L}_0)$.
- (c) For each $g \in N_{\mathcal{L}}(T)$, the restriction of c_g to T is an automorphism of \mathcal{E} , and one obtains in this way a homomorphism $\lambda_T : N_{\mathcal{L}}(T) \rightarrow \text{Aut}(\mathcal{E})$.

Proof. We check that (\mathcal{N}, Γ) is a localizable pair by verifying the two conditions in definition 1.1. The second of these - that T is maximal in the poset of p -subgroups of \mathcal{N} - is given by I.3.1(c).

Let $w \in \mathbf{W}(\mathcal{N})$ with $S_w \cap T \in \Gamma$. Then $X(S_w \cap T) \in \Delta$ by hypothesis, so $S_w \in \Delta$ and $w \in \mathbf{D}$. Thus the condition 1.1(1) obtains, and (\mathcal{N}, Γ) is a localizable pair. By 1.2 $(\mathcal{N}_\Gamma, \Gamma, T)$ is a locality. Since $\mathcal{E}^{cr} \subseteq \Gamma$ and \mathcal{E} is Γ -generated by hypothesis, 1.3 yields $\mathcal{E} = \mathcal{F}_T(\mathcal{N}_\Gamma)$.

Let $U \in \Gamma$ and set $P = XU$. Then $P \in \Delta$, so $N_{\mathcal{L}}(P)$ is a group of characteristic p . Then $N_{\mathcal{N}}(P)$ is of characteristic p by II.2.7(a), and then $N_{\mathcal{N}}(U)$ is of characteristic p by II.2.7(b). Thus $(\mathcal{N}_\Gamma, \Gamma, T)$ is a proper locality on \mathcal{E} .

We assume for the remainder of the proof that $O_p(\mathcal{L})\Gamma \subseteq \Delta$. Let $R \in \mathcal{F}^{cr}$ and, via II.1.18, let R' be an \mathcal{F} -conjugate of R such that both R' and $R' \cap T$ are fully normalized in \mathcal{F} , and then $R' \cap T \in \mathcal{E}^{cr}$ by 2.6(b). Then $O_p(\mathcal{N}) \leq R'$. Thus $R' \in \Delta_0$, and then also $R \in \Delta_0$. Thus $\mathcal{F}^{cr} \subseteq \Delta_0$, and the restriction $(\mathcal{L}_0, \Delta_0, T)$ of \mathcal{L} is then given by II.2.11 as a proper locality on \mathcal{F} . The equality $\mathcal{N}_\Gamma = \mathcal{N} \cap \mathcal{L}_0$ is immediate from the definitions of \mathcal{N}_Γ and of \mathcal{L}_0 . Thus (a) holds.

Set $H = N_{\mathcal{L}}(T)$. As $O_p(\mathcal{L})T = XT \in \Delta_0$, H is a subgroup of \mathcal{L}_0 . Let $x \in \mathcal{L}_0$, let $h \in H$, and set $P = S_x \cap O_p(\mathcal{L})T$. Then $P \in \Delta$ (by the definition of \mathcal{L}_0), and $(h^{-1}, x, h) \in \mathbf{D}$ via P^h . Points (b) and (c) now follow from 2.3. \square

The remainder of this section involves the following variation on the setup of 2.1.

2.9 (Product setup). (\mathcal{L}, Δ, S) is a proper locality on \mathcal{F} , and \mathcal{N}_1 and \mathcal{N}_2 are partial normal subgroups of \mathcal{L} . Set $T_i = S \cap \mathcal{N}_i$, and set $\mathcal{E}_i = \mathcal{F}_{T_i}(\mathcal{N}_i)$. Assume:

$$(*) \quad S \cap \mathcal{N}_i \leq C_S(\mathcal{N}_{3-i}) \quad (i = 1, 2).$$

Set $\mathcal{M} = \mathcal{N}_1\mathcal{N}_2$ (a partial normal subgroup of \mathcal{L} by I.5.1, or by [He]), and set $R = S \cap \mathcal{M}$ and $\mathcal{D} = \mathcal{F}_R(\mathcal{M})$.

Lemma 2.10. *Assume the setup of 2.9. Assume further that $\mathcal{D}^{cr} \subseteq \Delta$ and that \mathcal{D} is \mathcal{D}^{cr} -generated. Set $\Gamma = \{P \in \Delta \mid P \leq R\}$, and set*

$$\Gamma_i = \{PT_{3-i} \cap T_i \mid P \in \Gamma\}, \quad (i = 1, 2).$$

Let Δ_0 be the overgroup-closure of $\Gamma_1\Gamma_2$ in S , and let Δ_0^+ be the overgroup-closure of Γ in S . Then the following hold.

- (a) Δ_0 and Δ_0^+ are \mathcal{F} -closed subset of Δ , and $\mathcal{F}^{cr} \subseteq \Delta_0 \subseteq \Delta_0^+$.
- (b) (\mathcal{M}, Γ) and $(\mathcal{N}_i, \Gamma_i)$ ($i = 1, 2$) are localizable pairs, $(\mathcal{M}_\Gamma, \Gamma, R)$ is a proper locality on \mathcal{D} , and each $((\mathcal{N}_i)_{\Gamma_i}, \Gamma_i, T_i)$ is a proper locality on \mathcal{E}_i . Moreover, $(\mathcal{N}_i)_{\Gamma_i} = \mathcal{N}_i \cap \mathcal{M}_\Gamma$.
- (c) $R \in \Delta_0$, and the group $N_{\mathcal{L}}(R)$ acts by conjugation on all three of the localities in (b). Indeed, conjugation by $h \in N_{\mathcal{L}}(R)$ is an automorphism of \mathcal{D} which restricts to an automorphism of each \mathcal{E}_i .
- (d) $\mathcal{D}^{cr} = (\mathcal{E}_1)^{cr}(\mathcal{E}_2)^{cr}$.
- (e) Let $Y \leq R$ be a subgroup of R , set $Y_i = Y \cap T_i$, and suppose that $Y = Y_1Y_2$. Then $Y \in \mathcal{D}^c$ if and only if each Y_i is in $(\mathcal{E}_i)^c$, and $Y \in \mathcal{D}^s$ if and only if $Y_i \in (\mathcal{E}_i)^s$ ($i = 1, 2$).

Proof. By definition, \mathcal{D}^{cr} is \mathcal{D} -invariant, so Γ is \mathcal{D} -closed. Since \mathcal{D} is \mathcal{D}^{cr} -generated and $\mathcal{D}^{cr} \subseteq \Delta$, by hypothesis, we may appeal to 2.8 with \mathcal{M} in the role of \mathcal{N} . Thus $(\mathcal{M}_\Gamma, \bar{\Gamma})$ is a localizable pair, and $(\mathcal{M}_\Gamma, \bar{\Gamma}, R)$ is a proper locality on \mathcal{D} .

We have $R = R_1R_2 = T_1T_2$ by I.5.1, so $R \in \Delta_0$. For any $P \in \Gamma$ set $P_i = PT_{3-i} \cap T_i$. Then $P \leq P_1P_2$, and thus $\Gamma_1\Gamma_2 \subseteq \Gamma$. As $\Gamma \subseteq \Delta$ we then have $\Delta_0 \subseteq \Delta_0^+ \subseteq \Delta$. Clearly Γ is \mathcal{F} -invariant, so Δ_0^+ is \mathcal{F} -closed. For any $P = P_1P_2 \in \Gamma_1\Gamma_2$ and any $g \in \mathcal{L}$ with $P \leq S_g$ we have $P^g = (P_1)^g(P_2)^g \in \Gamma_1\Gamma_2$, and it follows that Δ_0 is \mathcal{F} -closed.

Set $H = N_{\mathcal{L}}(R)$. Then H is a subgroup of \mathcal{L} as $R \in \Gamma_1\Gamma_2$. Evidently Γ is H -invariant, so H acts on \mathcal{M}_Γ , and then H acts on \mathcal{D} by 2.3. Thus (b) and (c) hold insofar as these points refer only to \mathcal{M} and not to \mathcal{E}_i .

Let A and B be \mathcal{D} -conjugate subgroups of R , and let $\phi : A \rightarrow B$ be a \mathcal{D} -isomorphism. By definition, this means that there exists $w = (g_1, \dots, g_n) \in \mathbf{W}(\mathcal{M})$ such that ϕ may be written as a composition c_w of conjugation maps:

$$(*) \quad A = A_0 \xrightarrow{c_{g_1}} A_1 \rightarrow \dots \rightarrow A_{n-1} \xrightarrow{c_{g_n}} A_n = B.$$

By I.5.2 each g_k is a product $g_k = x_k y_k$ with $x_k \in \mathcal{N}_1$, $y_k \in \mathcal{N}_2$, and with $S_{g_k} = S_{(x_k, y_k)}$. Set

$$w' = (x_1, y_1, \dots, x_k, y_k), \quad u = (x_1, \dots, x_k), \quad \text{and} \quad v = (y_1, \dots, y_k).$$

Then $\phi = c_{w'}$, and then $\phi = c_u \circ c_v$ since $T_i \leq C_S(\mathcal{N}_{3-i})$. Thus:

- (1) Each \mathcal{D} -isomorphism may be factored as a composition $\psi_1 \circ \psi_2$, where ψ_i is a composition of conjugation maps by elements of \mathcal{N}_i ($i = 1, 2$).

As a consequence:

- (2) Let $X_i \leq T_i$ ($i = 1, 2$), and set $X = X_1X_2$. Then $X^{\mathcal{D}} = (X_1)^{\mathcal{E}_1}(X_2)^{\mathcal{E}_2}$. In particular, X is fully normalized in \mathcal{D} if and only if each X_i is fully normalized in \mathcal{E}_i .

Let $U_i \in (\mathcal{E}_i)^{cr}$ ($i = 1, 2$), and set $X = U_1U_2$. There then exists an \mathcal{E}_i -conjugate V_i of U_i such that V_i is fully normalized in \mathcal{E}_i , and such that $V_i = O_p(N_{\mathcal{E}_i}(V_i))$. Set $Y = V_1V_2$.

Then (2) shows that $Y \in X^{\mathcal{D}}$ and that Y is fully normalized in \mathcal{D} . We compute:

$$\begin{aligned} C_R(V_1 V_2) &= C_R(V_1) \cap C_R(V_2) = C_{T_1}(V_1) T_2 \cap C_{T_2}(V_2) T_1 \leq V_1 T_2 \cap V_2 T_1 \\ &= V_1 (T_2 \cap V_2 T_1) = V_1 V_2 (T_1 \cap T_2) \leq V_1 V_2, \end{aligned}$$

since $T_1 \cap T_2 \leq Z(T_1) \leq V_1$. Thus $V_1 V_2 \in \mathcal{D}^c$. In fact, we have shown:

$$(3) \quad (\mathcal{E}_1)^c (\mathcal{E}_2)^c \subseteq \mathcal{D}^c.$$

Let $\phi_1 : A \rightarrow B$ be an $N_{\mathcal{E}_1}(V_1)$ -isomorphism, where A and B are subgroups of $N_{T_1}(V_1)$ containing V_1 . Then ϕ_1 is a \mathcal{D} -isomorphism, and evidently ϕ extends to a \mathcal{D} -isomorphism $\gamma : AV_2 \rightarrow BV_2$ whose restriction to V_2 is the identity map. Set $P = O_p(N_{\mathcal{D}}(Y))$. As γ is in fact a $N_{\mathcal{D}}(Y)$ -isomorphism, γ extends to a \mathcal{D} -isomorphism $\psi : AP \rightarrow BP$ which leaves P invariant.

Write $\psi = \psi_1 \circ \psi_2$ as in (1). Thus,

$$AP \xrightarrow{\psi_1} CQ \xrightarrow{\psi_2} BP,$$

where $C = A\psi_1$ and $Q = P\psi_1 = P\psi_2^{-1}$. Then

$$Q \leq APT_1 \cap PT_2 = P(T_1 \cap T_2) = P,$$

since

$$T_1 \cap T_2 \leq Z(T_1) \cap Z(T_2) \leq V_1 \cap V_2 \leq V_1 V_2 \leq P.$$

This shows that $\psi_1 : AP \rightarrow BP$ is an extension of ϕ_1 . There is an obvious further extension of ψ_1 to a \mathcal{D} -isomorphism $\lambda_1 : APT_2 \rightarrow BPT_2$, given in the same way as ψ_1 as a composite of conjugations by elements of \mathcal{N}_1 . Thus ϕ_1 extends to an \mathcal{E}_1 -homomorphism $A(PT_2 \cap T_1) \rightarrow B(PT_2 \cap T_1)$. As $V_1 = O_p(N_{\mathcal{E}_1}(V))$ we conclude that $V_1 = PT_2 \cap T_1$. That is, V_1 is the projection of P into T_1 relative to the decomposition $R = T_1 T_2$. Similarly, V_2 is the projection of P into T_2 , and thus $P = V_1 V_2$. With (3), this shows:

$$(4) \quad \Delta^{cr} \supseteq (\mathcal{E}_1)^{cr} (\mathcal{E}_2)^{cr}.$$

Let P now be an arbitrary member of \mathcal{D}^{cr} . In order to show that $P \in (\mathcal{E}_1)^{cr} (\mathcal{E}_2)^{cr}$, it suffices to consider the case where P is fully normalized in \mathcal{D} , by (2). Set $M = N_{\mathcal{M}}(P)$. Then M is a normal subgroup of the group $N_{\mathcal{L}}(P)$. Set $P_i = PT_{3-i} \cap T_i$, and set $K_i = N_{\mathcal{N}_i}(P_i)$. Then $P \leq P_1 P_2 \trianglelefteq K_i$, and $P_1 P_2 \in \Delta$. Thus K_i is a normal subgroup of $N_{\mathcal{M}}(P_1 P_2)$. We again quote I.5.2 in order to write an arbitrary $g \in M$ as a product $g = x_1 x_2$ with $x_i \in \mathcal{N}_i$ and with $S_g = S_{(x_1, x_2)}$. Then $(P_i)^{x_i} = P_i$, and thus $M \leq K_1 K_2$. As \mathcal{M}_{Γ} is a proper locality, II.2.3 shows that $P = O_p(M)$. Then $N_{P_i}(P) \leq P$, and thus $P = P_1 P_2$. This completes the proof of (d).

In order to now show that $(\mathcal{N}_i, \Gamma_i)$ is a localizable pair, three points have to be verified. First: T_i is maximal in the poset of p -subgroups of \mathcal{N}_i (I.3.1(c)). Second: Γ_i is an \mathcal{E}_i -closed set of subgroups of T_i (true, since Γ_i is \mathcal{F} -invariant). Third: $w \in \mathbf{D}$ for each $w \in \mathbf{W}(\mathcal{N}_i)$ such that S_w contains some $U \in \Gamma_i$. This third point is a consequence of

(1) and (d), since $S_w \geq UT_{3-i} \in \Delta$. So then, $(\mathcal{N}_i, \Gamma_i)$ is a localizable pair, and then $((\mathcal{N}_i)_{\Gamma_i}, \Gamma_i, T_i)$ is a locality by 1.2.

Set $\mathcal{K}_i = \mathcal{N}_i \cap \mathcal{M}_\Gamma$ and set $\mathcal{D}_1 = \mathcal{F}_{T_1}(\mathcal{K}_1)$. Then $\mathcal{K}_i = (\mathcal{N}_i)_{\Gamma_i}$, and $(\mathcal{E}_1)^{cr} = (\mathcal{D}_1)^{cr}$, by (d). Thus $(\mathcal{E}_1)^{cr} \subseteq \Gamma_i$. As \mathcal{D} is \mathcal{D}^{cr} -generated, it follows from (1) and (d) that \mathcal{E}_1 is $(\mathcal{E}_1)^{cr}$ -generated. As \mathcal{D}_1 is a fusion subsystem of \mathcal{E}_1 , we conclude that $\mathcal{E}_1 = \mathcal{D}_1$. Since \mathcal{M}_Γ -normalizers of members of Γ are of characteristic p , the normalizer in \mathcal{K}_i of any member of Γ_i is of characteristic p , so $(\mathcal{N}_i)_{\Gamma_i}$ is a proper locality on \mathcal{E}_i . Thus (b) holds.

Evidently H acts on \mathcal{M}_Γ by conjugation, and this action restricts to an action on \mathcal{K}_i for each i . The proof of (c) is then completed by 2.8(c).

In order to complete the proof of (a) it remains to show that $\mathcal{F}^{cr} \subseteq \Delta_0$. Let $P \in \mathcal{F}^{cr}$ and let Q be an \mathcal{F} -conjugate of P such that both Q and $Q \cap R$ are fully normalized in \mathcal{F} . Then $Q \cap R \in \mathcal{D}^{cr}$ by 2.6(b). Then (d) shows that $Q \cap R \in \Delta_0$, so $Q \in \Delta_0$. As \mathcal{D}_0 is \mathcal{F} -invariant we obtain $P \in \Delta_0$, so (a) holds.

It now remains to prove (e). Let $Y = Y_1 Y_2$ be as stated in (e). By (2) we may proceed under the assumption that Y is fully normalized in \mathcal{F} and that each Y_i is fully normalized in \mathcal{E}_i . Suppose that Y is \mathcal{D} -centric. Then $C_{T_i}(Y_i) \leq C_S(Y) \cap T_i \leq Y \cap T_i = Y_i$, and thus $Y_i \in (\mathcal{E}_i)^c$. With (3), we then have the first “if and only if” in (e).

Set $Q_i = O_p(N_{\mathcal{E}_i}(Y_i))$ and set $Q = O_p(N_{\mathcal{D}}(Y))$. Then $Q_1 Q_2 = Q$, as follows from (1). Further, by II.1.16, Q is fully centralized in \mathcal{D} , and Q_i is fully centralized in \mathcal{E}_i . Since $Q_1 Q_2 \in \mathcal{D}^c$ if and only if each Q_i is centric in \mathcal{E}_i , we conclude that $Y \in \mathcal{D}^s$ if and only if each $Q_i \in (\mathcal{E}_i)^s$. \square

Corollary 2.11. *Assume the setup of 2.9, with $\mathcal{N}_1 \mathcal{N}_2 = \mathcal{L}$. Set $\Gamma_i = \{PT_{3-i} \cap T_i \mid P \in \Delta\}$ ($i = 1, 2$). Then $(\mathcal{N}_i, \Gamma_i, T_i)$ is a proper locality on \mathcal{E}_i , $\Gamma_1 \Gamma_2$ is an \mathcal{F} -closed subset of Δ , and \mathcal{L} is the same partial group as its restriction $\mathcal{L}|_{\Gamma_1 \Gamma_2}$ to $\Gamma_1 \Gamma_2$.*

Proof. Since \mathcal{F} is (cr) -generated by II.2.3, and since $\mathcal{F}^{cr} \subseteq \Delta$, we may apply 2.10 with $\mathcal{M} = \mathcal{L}$ and with $\Gamma = \Delta$. Then $(\mathcal{N}_i, \Gamma_i, T_i)$ is a proper locality on \mathcal{E}_i , by 2.10(b), and $\mathcal{L}_\Gamma = \mathcal{L}$. Set $\Delta_0 = \Gamma_1 \Gamma_2$. Then Δ_0 is \mathcal{F} -closed, and $\mathcal{F}^{cr} \subseteq \Delta_0 \subseteq \Delta$ by 2.10(a). Let $g \in \mathcal{L}$ and write $g = g_1 g_2$ with $g_i \in \mathcal{N}_i$ and with $S_g = S_{(g_1, g_2)}$ (via I.5.2). Then $S_g = (S_{g_1} \cap T_1)(S_{g_2} \cap T_2) \in \Delta_0$, so $\mathcal{L}_{\Delta_0} = \mathcal{L}$. \square

Section 3: Alperin-Goldschmidt variations

For any finite group G let $\Gamma_p(G)$ be the graph whose vertices are the Sylow p -subgroups of G , and whose edges are the pairs $\{X, Y\}$ of distinct Sylow p -subgroups such that $X \cap Y \neq 1$. By Sylow’s theorem, the action of G on $\Gamma_p(G)$ by conjugation is transitive on vertices.

Let S be a fixed Sylow p -subgroup of G , let Σ be the connected component of $\Gamma_p(G)$ containing the vertex S , and let H be the set-wise stabilizer of Σ in G . Then $g \in H$ for all $g \in G$ such that p divides $|H \cap H^g|$, and hence $H = G$ if and only if $\Gamma_p(G)$ is connected. We shall say that G is p -disconnected if $\Gamma_p(G)$ is disconnected. Otherwise, G is p -connected.

Remark. In the standard terminology, one says that a proper subgroup X of G such that p divides $|X|$ and such that $g \in K$ whenever p divides $|X \cap X^g|$ is *strongly p -embedded* in G . One easily deduces in that case, that $\text{Syl}_p(X)$ contains a connected component Σ of $\Gamma_p(G)$, and that X contains the set-wise stabilizer H of Σ . Thus G has a strongly p -embedded subgroup if and only if $\Gamma_p(G)$ is disconnected.

The following result is well known. Since it is not so easy to find a reference for it, we provide a proof.

Lemma 3.1. *Let X be a p' -group and let A be an elementary abelian group of order p^2 such that A acts on X . Then $X = \langle C_X(a) \mid 1 \neq a \in A \rangle$.*

Proof. Let G be the semi-direct product XA formed via the action of A on X . Let q be a prime dividing $|X|$ and let Y be a Sylow q -subgroup of X . Then $G = N_G(Y)X$, so we may choose Y to be A -invariant. Then A acts on the elementary abelian q -group $V := Y/\Phi(Y)$. By Maschke's Theorem V is a direct sum of irreducible A -submodules. By Schur's Lemma $C_A(W)$ contains a maximal subgroup of A for each irreducible A -submodule W of V . As A is assumed to be non-cyclic we thereby obtain $V = \langle C_V(a) \mid 1 \neq a \in A \rangle$. Here $C_V(a) = C_Y(a)\Phi(Y)/\Phi(Y)$ by coprime action (cf. II.2.7(c)), so $Y = \langle C_Y(a) \mid 1 \neq a \in A \rangle\Phi(Y)$. It is a basic property of the Frattini subgroup that if Y_0 is a subgroup of Y such that $Y = Y_0\Phi(Y)$ then $Y = Y_0$. Thus we have the lemma in the case that $X = Y$. We conclude that $\langle C_X(a) \mid 1 \neq a \in A \rangle$ contains a Sylow subgroup of X for each prime divisor of $|X|$, and this completes the proof. \square

Lemma 3.2. *Let G be a p -disconnected finite group, and let \mathbb{K} be the poset (via inclusion) of normal subgroups K of G such that p divides $|K/O_p(G)|$ and such that $O_p(G) \leq K$. Then there exists a unique minimal $K \in \mathbb{K}$.*

Proof. We may assume that $O_p(G) = 1$. Fix a Sylow p -subgroup S of G and a strongly p -embedded subgroup H of G containing S . Suppose that there exist K, K' minimal in \mathbb{K} with $K \neq K'$, and set $X = K \cap K'$. Then X is a p' -group, and $S \cap KK'$ contains an elementary abelian subgroup A of order p^2 . By 3.1, X is then generated by its subgroups $C_X(a)$ as a varies over the set of non-identity elements of $S \cap KK'$. Thus $X \leq H$. Since $[K, S \cap K'] \leq X$ we have $K \leq N_G(S \cap K')X$, and so $K \leq H$. Then $G = N_G(S \cap K)K \leq H$, which is contrary to H being a proper subgroup of G . Thus \mathbb{K} has a unique minimal member K . \square

In what follows we shall refer to the group K in the preceding lemma as the *p -socle* of G .

Notation 3.3. Let (\mathcal{L}, Δ, S) be a proper locality on \mathcal{F} , and let $T \leq S$ be strongly closed in \mathcal{F} . Denote by $\mathbf{A}(\mathcal{F})$ the set of all $P \in \mathcal{F}^{cr}$ such that:

- (1) $N_S(P) \in \text{Syl}_p(N_{\mathcal{L}}(P))$, and
- (2) either $P = S$ or $N_{\mathcal{L}}(P)/O_p(N_{\mathcal{L}}(P))$ is p -disconnected.

Let $\mathbf{A}_T(\mathcal{F})$ be the set of all $P \in \mathbf{A}(\mathcal{F})$ such that also:

- (3) $P \cap T$ is fully normalized in \mathcal{F} .

Note that the condition (1) is equivalent to the statement that P is fully normalized in \mathcal{F} , and that condition (2) is equivalent to the statement that either $P = S$ or $\text{Out}_{\mathcal{F}}(P)$ is p -disconnected. Thus the sets $\mathbf{A}(\mathcal{F})$ and $\mathbf{A}_T(\mathcal{F})$ depend only on \mathcal{F} and T .

Definition 3.4. An element $g \in \mathcal{L}$ is $\mathbf{A}_T(\mathcal{F})$ -decomposable if there exists $w \in \mathbf{D}$ and a sequence σ of members of $\mathbf{A}_T(\mathcal{F})$:

$$w = (g_1, \dots, g_n), \quad \sigma = (P_1, \dots, P_n),$$

such that the following hold.

- (1) $S_g = S_w$ and $g = \Pi(w)$.
- (2) $P_i = S_{g_i}$ for all i .
- (3) Either $P_i = S$ or $g_i \in O^p(K_i)$, where K_i is the p -socle of $N_{\mathcal{L}}(R_i)$.

We say also that (w, σ) is an $\mathbf{A}_T(\mathcal{F})$ -decomposition of g if (1) through (3) hold.

Since condition (2) in 3.4 implies that the sequence σ is determined by w , there is some redundancy in the definition. For that reason we shall also speak of the $\mathbf{A}_T(\mathcal{F})$ -decomposition w and its *auxiliary sequence* σ .

The following result is a version of the Alperin-Goldschmidt fusion theorem [Gold].

Theorem 3.5. *Let (\mathcal{L}, Δ, S) be a proper locality on \mathcal{F} , and let T be strongly closed in \mathcal{F} . Then every element of \mathcal{L} is $\mathbf{A}_T(\mathcal{F})$ -decomposable.*

Proof. Set $\mathbf{A} = \mathbf{A}_T(\mathcal{F})$. The following point is then immediate from definition 3.4.

- (1) Let $u = (x_1, \dots, x_k) \in \mathbf{D}$ with $S_u = S_{\Pi(u)}$. Suppose that each x_i is \mathbf{A} -decomposable. Then $\Pi(u)$ is \mathbf{A} -decomposable.

Among all $g \in \mathcal{L}$ such that g is not \mathbf{A} -decomposable, choose g with $|S_g|$ as large as possible, and set $P = S_g$. Then $P \neq S$, as otherwise $w = (g)$ and $\sigma = (S)$ provide an \mathbf{A} -decomposition for g . Set $P' = P^g$.

As T is strongly closed in \mathcal{F} , there exists an \mathcal{F} -conjugate Q of P (and hence also of P') such that both Q and $Q \cap T$ are fully normalized in \mathcal{F} , by II.1.15. As $P, P' \in \Delta$, there exist $a, b \in \mathcal{L}$ such that $P^a = Q$, $(P')^b = Q$, and $N_S(P)^a \leq N_S(Q) \geq N_S(P')^b$. The maximality of $|P|$ implies that a and b are \mathbf{A} -decomposable, and the same is then true of a^{-1} and b^{-1} via the inverses of the words (and the reversals of the sequences of subgroups of S) which yield \mathbf{A} -decomposability for a and b . Set $g' = a^{-1}gb$ and set $M = N_{\mathcal{L}}(Q)$. Then $g' \in M$, $(a, g', b^{-1}) \in \mathbf{D}$ via Q , and $ag'b^{-1} = g$. If g' has an \mathbf{A} -decomposition then so does g , by (1). Thus we may assume that $g = g'$, whence $P = Q = P'$.

Set $R = N_S(Q)$. Then $R \in \text{Syl}_p(M)$ by II.2.1, so $O_p(M) \leq S$, and then $O_p(M) = Q$. Let K be a normal subgroup of M which is minimal subject to $Q \leq K$ and $Q \notin \text{Syl}_p(K)$. By the Frattini lemma we may write $g = fh$, where $f \in O^p(K)$ and $h \in N_M(S \cap K)$. Then h is \mathbf{A} -decomposable, and $S_{(f,h)} = S_g$, so (1) implies that f has no \mathbf{A} -decomposition. Thus we may assume that $g \in O^p(K)$. If M/Q is p -disconnected then K is the p -socle of M , so $Q \in \mathbf{A}$, and $((g), (Q))$ is a \mathbf{A} -decomposition of g . Thus M/Q is p -connected.

Let Γ be the graph $\Gamma_p(M/Q)$, and let $R = R_0, \dots, R_m = R^g$ be a sequence of Sylow p -subgroups of M such that $(R_0/Q, \dots, R_m/Q)$ is a geodesic path in Γ from R/Q to

R^g/Q . We may assume that, among all $g \in K$ having no \mathbf{A} -decomposition, g has been chosen so that the distance m from R/Q to R^g/Q in Γ is as small as possible. Then $m \neq 0$ as $R \not\leq Q$. Suppose $m = 1$. Then $R \cap R^g$ properly contains Q , and then $(R \cap R^g)^{g^{-1}}$ is a subgroup of S_g which properly contains Q , contrary to $S_g = Q$. Thus $m \geq 2$.

Let $d \in M$ such that $(R_{m-1})^d = R_m$, and set $h = gd^{-1}$. Then $R^h = R_{m-1}$, and the minimality of m implies that there exists an \mathbf{A} -decomposition u of h . Since $g = hd$ we have $Q = S_{(h,d)}$, and then (1) implies that d has no \mathbf{A} -decomposition. Then $S_d = Q$ by the maximality of $|S_g|$ in our choice of g . Set $e = h d h^{-1}$. Conjugation by h sends the pair (R, R^e) to (R_{m-1}, R_m) , so R/Q is adjacent to R^e/Q in Γ . As $m > 1$ there then exists an \mathbf{A} -decomposition v for e . Set $w = u^{-1} \circ v \circ u$. Then $\Pi(w) = h^{-1} e h = d$, and then w is an \mathbf{A} -decomposition for d since $S_w \geq Q = S_d$. But we have already determined that d has no \mathbf{A} -decomposition, and this contradiction completes the proof. \square

We shall assume the setup of 2.1 until introducing a variation on that setup in 2.9. Thus, for now, (\mathcal{L}, Δ, S) is a proper locality on \mathcal{F} , $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup, $T = \mathcal{N} \cap S$, and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$.

Lemma 3.6. *Let $P \in \mathbf{A}(\mathcal{F})$, set $M = N_{\mathcal{L}}(P)$, and set $U = P \cap T$. If $P \neq S$, let K be the p -socle of M .*

- (a) *If $P \in \mathbf{A}_T(\mathcal{F})$ then $U^{\mathcal{F}} \subseteq \mathcal{E}^c$, and $U = O_p(N_{\mathcal{E}}(U))$.*
- (b) *Either $T \leq P$ or $C_S(U) \leq P$; and if $T \not\leq P$ then $K \leq N_{\mathcal{N}}(P)P$.*

Proof. Point (a) is a direct application of 2.6(b). If $P = S$ then $U = T$, and then (b) holds vacuously. Thus we may assume that $P \neq S$, and hence that M is p -disconnected. Set $K = N_{\mathcal{N}}(P)$. Then $K \trianglelefteq M$, and $[P, K] \leq P \cap K \leq U$.

Set $D = N_{C_S(U)}(P)$ and set $X = C_S(U)P$. Then $N_X(P) = DP$. Set $H = \langle D^M \rangle$. Then $[U, H] = 1$, so $H \cap K$ centralizes the chain $P \geq U \geq 1$. As M is of characteristic p , II.2.7(c) yields $H \cap K \leq P$. Thus HKP/P is the direct product of HP/P with KP/P . As M is p -disconnected, either HP/P or KP/P is a p' -group, so either $N_T(P) \leq P$ or $D \leq P$. This yields the first of the statements in (b). Now suppose that $T \not\leq P$. Then $N_T(P) \not\leq P$, and hence $K \leq \langle N_T(P)^M \rangle P \leq N_{\mathcal{N}}(P)P$. Thus (b) holds. \square

Set $H = N_{\mathcal{L}}(C_S(T)T)$. Then H is a subgroup of \mathcal{L} , by 2.2(a).

Definition 3.7. Denote by $\Sigma_T(\mathcal{F})$ the set of all products $O_p(\mathcal{L})UV$ such that:

- $U = (P \cap T)^h$ for some $P \in \mathbf{A}_T(\mathcal{F})$ and some $h \in H$, and
- $V \in C_{\mathcal{F}}(T)^{cr}$.

Recall from 1.5 that $(N_{\mathcal{L}}(T), \Delta, S)$ is a proper locality on $N_{\mathcal{F}}(T)$, and that $C_{\mathcal{L}}(T) \trianglelefteq N_{\mathcal{L}}(T)$. We shall often write \mathcal{L}_T for the locality $N_{\mathcal{L}}(T)$, and \mathcal{C}_T for $C_{\mathcal{L}}(T)$.

Lemma 3.8. *Let $X \in \Sigma_T(\mathcal{F})$, and write $X = O_p(\mathcal{L})UV$ as in the preceding definition. Then $U = X \cap T$ and $V = C_X(T)$. Moreover, $\Sigma_T(\mathcal{F})$ depends only on \mathcal{F} and T , and not on \mathcal{L} .*

Proof. Notice first of all that $O_p(\mathcal{L})T \leq O_p(\mathcal{L}_T)$. As \mathcal{L}_T is a proper locality on $N_{\mathcal{F}}(T)$ we have $O_p(\mathcal{L}_T) = O_p(N_{\mathcal{F}}(T))$, and then

$$O_p(\mathcal{L})T \cap \mathcal{C}_T \leq O_p(N_{\mathcal{F}}(T)) \cap \mathcal{C}_T \leq O_p(\mathcal{F}_{C_S(T)}(\mathcal{C}_T)) \leq O_p(C_{\mathcal{F}}(T)),$$

since $\mathcal{F}_{C_S(T)}(\mathcal{C}_T)$ is a subsystem of $C_{\mathcal{F}}(T)$. Then

$$(*) \quad C_X(T) = C_{O_p(\mathcal{L})U}(T)V \leq O_p(C_{\mathcal{F}}(T))V = V,$$

since $V \in C_{\mathcal{F}}(T)^{cr}$.

Let $P \in \mathbf{A}_T(\mathcal{F})$ and let $h \in H$ with $U = (P \cap T)^h$. If $T \leq P$ then $U = T = X \cap T$. So assume that $T \not\leq P$. Then $C_S(T) \leq P$ by 3.6. As $P \in \mathcal{F}^{cr}$ we have also $O_p(\mathcal{L}) \leq P$. Then

$$X^{h^{-1}} = O_p(\mathcal{L})(P \cap T)V^{h^{-1}} \leq P,$$

so $X^{h^{-1}} \cap T = P \cap T$, and $X \cap T = U$.

Since $O_p(\mathcal{L}) = O_p(\mathcal{F})$, and since H -conjugation on subgroups of T is the same as $\text{Aut}_{\mathcal{F}}(C_S(T)T)$ -conjugation (2.2(c)), $\mathcal{F}_T(\mathcal{F})$ depends only on \mathcal{F} and T . \square

Lemma 3.9. *Assume $\Sigma_T(\mathcal{F}) \subseteq \Delta$. Then:*

- (a) $\mathcal{F}_{C_S(T)}(\mathcal{C}_T) = C_{\mathcal{F}}(T)$.
- (b) *The image of H under the natural homomorphism $H \rightarrow \text{Aut}(C_S(T))$ is contained in $\text{Aut}(C_{\mathcal{F}}(T))$.*
- (c) *Every member of $\Sigma_{C_S(T)}(N_{\mathcal{F}}(T))$ contains a member of $S_T(\mathcal{F})$. In particular, $\Sigma_{C_S(T)}(N_{\mathcal{F}}(T)) \subseteq \Delta$.*

Proof. Set $\Psi = \{Y \leq C_S(T) \mid O_p(\mathcal{L})TY \in \Delta\}$. Since $\Sigma_T(\mathcal{F}) \subseteq \Delta$ we have $C_{\mathcal{F}}(T)^{cr} \subseteq \Psi$. We may then appeal to 1.4(c), with $O_p(\mathcal{L})T$ in the role of V and with Ψ in the role of Σ . Thus (\mathcal{C}_T, Ψ) is a localizable pair and $(\mathcal{C}_T)_{\Psi}, \Psi, C_S(T))$ is a proper locality on $\mathcal{C}_{\mathcal{F}}(T)$. This yields (a).

Set $\mathcal{F}_T = N_{\mathcal{F}}(T)$ and let Δ_0 be the \mathcal{F}_T -closure of $\Sigma_T(\mathcal{F})$. Let $R \in (\mathcal{F}_T)^{cr}$. As $\mathcal{F}_T = \mathcal{F}_S(\mathcal{L}_T)$, \mathcal{F}_T is inductive by 6.1. By II.1.15 there is then an \mathcal{F}_T -conjugate R' of R such that $R' \cap C_S(T)$ is fully normalized in \mathcal{F}_T . Then $R' \cap C_S(T) \in C_{\mathcal{F}}(T)^{cr}$ by 2.6(b). Since $T \leq R'$, we have thus shown that $(\mathcal{F}_T)^{cr} \subseteq \Delta_0$. Since $\Delta_0 \subseteq \Delta$, we have the restriction \mathcal{L}_0 of \mathcal{L} to Δ_0 (II.1.10), and plainly H is a subgroup of \mathcal{L}_0 . Then 2.3 applies with $(\mathcal{C}_T)_{\Psi}$ in the role of \mathcal{N} , and yields the desired action of H on $C_{\mathcal{F}}(T)$. That is, (b) holds.

Set $T^* = C_S(C_S(T))$. By definition 3.8, $\Sigma_{C_S(T)}(\mathcal{L}_T)$ is the set of products $O_p(\mathcal{L}_T)UV$ such that:

- (1) $U = (Q \cap C_S(T))^h$ for some $Q \in \mathbf{A}_{C_S(T)}(\mathcal{L}_T)$ and some $h \in N_{\mathcal{L}_T}(T^*)$, and
- (2) $V \in C_{\mathcal{F}_T}(C_S(T))^{cr}$.

Let h, U , and V be as in (1) and (2). Then $h \in H := N_{\mathcal{L}}(T)$, and (2) yields $T \leq V$. Set $U' = Q \cap C_S(T)$. Then $U' \in C_{\mathcal{F}}(T)^{cr}$ by 2.6(b), and then (b) yields $U \in C_{\mathcal{F}}(T)^{cr}$. Then $O_p(\mathcal{L})TU \in \Sigma_T(\mathcal{F})$. Since $TU \leq VU$ we then obtain (c). \square

Lemma 3.10. *Let $U \in \mathcal{E}^c$ and let V be a subgroup of $C_S(T)$, such that $O_p(\mathcal{L})UV \in \Delta$. Then $N_{\mathcal{N}}(C_S(T)U)$ and $N_{\mathcal{C}_T}(V)$ are subgroups of the group $N_{\mathcal{L}}(UV)$, and*

$$[N_{\mathcal{N}}(C_S(T)U), N_{\mathcal{C}_T}(V)] = [N_{\mathcal{N}}(UV), O^p(N_{\mathcal{C}_T}(UV))] = 1.$$

Proof. Set $R = UV$ and set $M^* = N_{\mathcal{L}}(O_p(\mathcal{L})R)$, $M = N_{\mathcal{L}}(R)$, $K = N_{\mathcal{N}}(R)$, $K_0 = N_{\mathcal{N}}(C_S(T)U)$, and $X = N_{\mathcal{C}_T}(R)$. As $O_p(\mathcal{L})R \in \Delta$, M^* is a subgroup of \mathcal{L} , and $M = N_{M^*}(R)$ is a subgroup of \mathcal{L} . Moreover, M and K are of characteristic p by II.2.6.

We have $[C_S(T), K_0] \leq C_S(T) \cap \mathcal{N} = Z(T) \leq U$, as $U \in \mathcal{E}^c$. Thus $K_0 \leq K$, while clearly $N_{\mathcal{C}_T}(V) \leq X$. As K is of characteristic p and $U \in \mathcal{E}^c$ we have $C_K(U) \leq U$. Then $[K, X] \leq U$, and so $[K, O^p(X)] = 1$ by coprime action. \square

Definition 3.11. Let $g \in \mathcal{L}$, and let $w = (a, g_1, \dots, g_n) \in \mathbf{D}$. Then w is an \mathcal{N} -decomposition of g if:

- (1) $S_w = S_g$,
- (2) $\Pi(w) = g$,
- (3) $a \in N_{\mathcal{L}}(T)$, and
- (4) there exists a sequence (X_1, \dots, X_n) of members of $\Sigma_T(\mathcal{F}) \cap \Delta$ such that $C_S(T) \leq X_i$ and such that

$$g_i \in O^p(N_{\mathcal{N}}(X_i)) \cap O^{p'}(N_{\mathcal{N}}(X_i)) \quad (1 \leq i \leq n).$$

A word of caution regarding the above definition: Even though $C_S(T) \leq X_i$ for all i , it cannot be concluded that $C_S(T) \leq S_w$. Indeed, there are examples to the contrary. The point is that $C_S(T)$ need not be invariant under g_i .

The following result may be thought of as a refinement of the splitting Lemma (I.3.12).

Theorem 3.12. *Assume $\Sigma_T(\mathcal{F}) \subseteq \Delta$. Then each $g \in \mathcal{L}$ has an \mathcal{N} -decomposition. Moreover, if $g \in \mathcal{N}$, and $w = (a, g_1, \dots, g_n)$ is an \mathcal{N} -decomposition of g , then $a \in N_{\mathcal{N}}(C_S(T)T)$.*

Proof. Set $\tilde{\Sigma} = \{X \in \Sigma_T(\mathcal{F}) \mid C_S(T) \leq X\}$. Let $g \in \mathcal{L}$, and let Φ be the set of all words $v = (x_1, \dots, x_n) \in \mathbf{W}(\mathcal{L})$ such that $S_w = S_g$, $\Pi(w) = g$, and having the following property.

- (*) For each i , one of the following holds.
 - (i) $x_i \in H$, and $S_{x_i} \in \mathbf{A}(N_{\mathcal{F}}(T))$.
 - (ii) $C_S(T) \not\leq S_{x_i} \in \mathbf{A}(N_{\mathcal{F}}(T))$, and $x_i \in O^p(K_i)$ where K_i is the p -socle of $N_{\mathcal{L}}(S_{x_i})$.
 - (iii) $T \not\leq S_{x_i}$, and $x_i \in O^p(N_{\mathcal{N}}(X_i)) \cap O^{p'}(N_{\mathcal{N}}(X_i))$ for some $X_i \in \tilde{\Sigma}$.

We shall see, first of all, that Φ contains the set of all $\mathbf{A}_T(\mathcal{F})$ -decompositions of g . Let $u = (x_1, \dots, x_n)$ be an arbitrary such. By 3.6, (i) or (ii) holds for any index j such that $S_{x_j} \in \mathbf{A}(\mathcal{L}_T)$. Let then i be an index such that $S_{x_i} \notin \mathbf{A}(\mathcal{L}_T)$. Set $x = x_i$, $P = S_{x_i}$,

and let K be the p -socle of $N_{\mathcal{L}}(P)$. Then $K = O^{p'}(K)$ by definition, and $K \leq N_{\mathcal{N}}(P)P$ by 3.6(b). Set $D = (P \cap T)C_S(T)$. Then $D \in \tilde{\Sigma}$, and $D \leq P$ (again by 3.6(b)). Then $D \trianglelefteq K$ since $[P, N_{\mathcal{N}}(P)] \leq P \cap T$. Since $x \in O^p(K)$ by definition 3.4, we obtain (iii), and thus $u \in \Phi$.

Now let v be an arbitrary member of Φ , and suppose that there is segment (c, b) of v such that $T \not\leq S_c$ and $T \leq S_b$. Let $X \in \tilde{\Sigma}$ such that $c \in N_{\mathcal{L}}(X)$ (and such that c and X in the role of x_i and X_i satisfy also the stronger condition given by (iii) in (*)).

Set $Q = S_b$, and suppose first that $C_S(T) \not\leq Q$. Then $b \in O^p(L)$, where L is the p -socle of $N_{\mathcal{L}}(Q)$. Set $E = N_{C_S(T)}(Q)$. Then $E \not\leq Q$, so 2.2 yields $L \leq \langle E^{N_{\mathcal{L}}(Q)} \rangle$, and hence $L \leq C_{\mathcal{L}}(T)$ as $T \trianglelefteq L$. Let $X \in \tilde{\Sigma}$ such that $c \in N_{\mathcal{L}}(X)$ (and such that c and X in the role of x_i and X_i satisfy also the stronger condition given by (iii)). We have $C_Q(T) \in C_{\mathcal{F}}(T)^{cr}$ by 2.6(b), so $[K, O^p(L)] = 1$ by 3.10. Thus c and b commute. Since $c \in \mathcal{N}$ and $b \in N_{\mathcal{L}}(T)$ we have $S_{(c,b)} = S_{(b,c^b)}$ by I.3.2(a), and so $S_{(c,b)} = S_{(b,c)}$. Write $v = v_1 \circ (c, b) \circ v_2$, and set $v' = v_1 \circ (b, c) \circ v_2$. Then $S_v = S_{v'}$ (so $v' \in \mathbf{D}$), and $\Pi(v') = \Pi(v)$. The condition (*) is evidently in place for v' , so $v' \in \Phi$.

Suppose next that $C_S(T) \leq Q$. Then $C_S(T)T \trianglelefteq N_{\mathcal{L}}(Q)$, and so $b \in H$. We have $X = (X \cap T)C_S(T)$ by the definition of $\tilde{\Sigma}_T(\mathcal{F})$, and $X \in \Delta$ by hypothesis. Then $(b, b^{-1}, c, b) \in \mathbf{D}$ via X , and $S_{(c,b)} = S_{(b,c^b)}$ as before. Conjugation by b is an isomorphism from $N_{\mathcal{L}}(X)$ to $N_{\mathcal{L}}(X^b)$ (I.2.3(b)), so $c^b \in O^p(N_{\mathcal{L}}(X^b))$ and $c^b \in O^{p'}(N_{\mathcal{L}}(X^b))$. Moreover we have $X^b \in \tilde{\Sigma}$, since $\tilde{\Sigma}$ is H -invariant. With v defined as in the preceding case, set $v' = v_1 \circ (b, c^b) \circ v_2$. We conclude that $v' \in \Phi$.

It follows from the preceding analysis that there exists $v \in \Phi$ such that v can be written as $v = v_1 \circ v_2$, where (i) or (ii) holds for each entry of v_1 , and where (iii) holds for each entry of v_2 . As v_1 is a prefix of a member of \mathbf{D} we have $v_1 \in \mathbf{D}$. Set $a = \Pi(v_1)$. Then $a \in \mathcal{L}_T$, and set $w = (a) \circ v_2$. Then $\Pi(w) = g$, and since $S_g \leq S_w \leq S_v = S_g$ we obtain $S_g = S_w$, and w is an \mathcal{N} -decomposition of g .

Suppose finally that $g \in \mathcal{N}$. Then $a \in \mathcal{N}$ by cancellation (I.1.4(e)), and so $a \in N_{\mathcal{N}}(T)$. As \mathcal{L} is proper, it follows from I.3.5 that $N_{\mathcal{N}}(T) \leq H$. Thus $a \in H$, and the proof is complete. \square

Corollary 3.13. *Assume that $\Sigma_T(\mathcal{F}) \subseteq \Delta$, and let Γ be the set of subgroups U of T of the form $(P \cap T)^h$, with $P \in \mathbf{A}_T(\mathcal{F})$ and with $h \in H$. Then $\Gamma \subseteq \mathcal{E}^c$, and \mathcal{E} is Γ -generated.*

Proof. The existence of an \mathcal{N} -decomposition for $g \in \mathcal{N}$ shows that the conjugation map $c_g : S_g \cap T \rightarrow (S_g \cap T)^g$ is a composition of \mathcal{E} -homomorphisms between members of Γ . Thus \mathcal{E} is Γ -generated. That Γ is a subset of \mathcal{E}^c is given by 2.6(b). \square

The following corollary to 3.12 answers a question that was left hanging from Part II. Namely, in Theorem A (the union of Theorems A1 and A2) one has invariance of the poset of partial normal subgroups of \mathcal{L} under expansion of objects from Δ to Δ^+ , but nothing is said about what becomes of the fusion systems of the various partial normal subgroups of \mathcal{L} under this process. One is told only (in the setup of Theorem A2) that if $\mathcal{N} \trianglelefteq \mathcal{L}$ with $T = S \cap \mathcal{N}$, then also $T = S \cap \mathcal{N}^+$.

Corollary 3.14. *Assume that $\Sigma_T(\mathcal{F}) \subseteq \Delta$, and let Δ^+ be an \mathcal{F} -closed set of subgroups of S such that $\mathcal{F}^{cr} \subseteq \Delta \subseteq \Delta^+ \subseteq \mathcal{F}^s$. Let $(\mathcal{L}^+, \Delta^+, S)$ be the unique (in the sense of Theorem I.A1) proper locality on \mathcal{F} which contains \mathcal{L} and such that the inclusion map $\mathcal{L} \rightarrow \mathcal{L}^+$ is a homomorphism of partial groups. Let \mathcal{N}^+ be the partial normal subgroup of \mathcal{L}^+ (as in Theorem II.A2) generated as a partial subgroup of \mathcal{L}^+ by the set of all \mathcal{L}^+ -conjugates of elements of \mathcal{N} . Then:*

- (a) $\mathcal{F}_T(\mathcal{N}) = \mathcal{F}_T(\mathcal{N}^+)$, and
- (b) \mathcal{N}^+ is the partial subgroup $\langle \mathcal{N} \rangle$ of \mathcal{L}^+ generated by \mathcal{N} .

Proof. By 3.12, \mathcal{E} is generated by the set of conjugation maps $c_g : S_g \cap T \rightarrow T$ such that $g \in \mathcal{N}$ and such that S_g contains a member of $\tilde{\Sigma}_T(\mathcal{F})$. The same is true of \mathcal{E}^+ , so (a) holds.

Let $g \in \mathcal{N}^+$, and let w be an \mathcal{N}^+ -decomposition of g . Then w is a sequence of elements of \mathcal{N} , and so $\mathcal{N}^+ \leq \langle \mathcal{N} \rangle$. The reverse inclusion is immediate (as \mathcal{N}^+ is a partial subgroup of \mathcal{L}^+). Thus (b) holds. \square

Lemma 3.15. *Suppose that $O_p(\mathcal{L})\mathcal{E}^c \subseteq \Delta$. Then:*

- (a) $\Sigma_T(\mathcal{F}) \subseteq \Delta$.
- (b) *The image of the natural homomorphism $\mathcal{L}_T \rightarrow \text{Aut}(T)$ is contained in $\text{Aut}(\mathcal{E})$.*
- (c) $(\mathcal{N}, \mathcal{E}^c)$ is a localizable pair, and $((\mathcal{N})_{\mathcal{E}^c}, \mathcal{E}^c, T)$ is a proper locality on \mathcal{E} .

Proof. Let $P \in \mathbf{A}_T(\mathcal{F})$, and set $U = P \cap T$. Then $U^H \subseteq \mathcal{E}^c$ by 2.6(b), and then $O_p(\mathcal{L})U^H C_{\mathcal{F}}(T)^{cr} \subseteq \Delta$ by hypothesis. This establishes (a). Set $\Gamma = \mathcal{E}^c$. Then 3.12 shows that \mathcal{E} is Γ -generated.

Let $V \in \Delta$ with $P \leq T$, set $M = N_{\mathcal{L}}(V)$, and set $K = N_{\mathcal{N}}(V)$. Then K is of characteristic p by II.2.6(a), and then $U \in \mathcal{E}^c$ if and only if $C_K(V) = Z(V)$. This shows that \mathcal{E}^c is \mathcal{F} -invariant.

As $\Gamma \subseteq \Delta$, and since T is maximal in the poset of p -subgroups of \mathcal{N} , it is immediate that (\mathcal{N}, Γ) is a localizable pair. Then $(\mathcal{N}_{\Gamma}, \Gamma, T)$ is a locality on a fusion subsystem \mathcal{E}_0 of \mathcal{E} , by 1.2. As \mathcal{E} is Γ -generated we get $\mathcal{E}_0 = \mathcal{E}$, and then (c) follows from 1.3. Point (b) is given by 2.3. \square

We end this section with an application of 3.5.

Theorem 3.16. *Let (\mathcal{L}, Δ, S) be a proper locality on \mathcal{F} , and let \mathbb{K} be the set of all partial normal subgroups $\mathcal{K} \trianglelefteq \mathcal{L}$ such that \mathcal{L}/\mathcal{K} is an abelian group. Set $[\mathcal{L}, \mathcal{L}] = \bigcap \mathbb{K}$. Then:*

- (a) $[\mathcal{L}, \mathcal{L}] = \langle [N_{\mathcal{L}}(P), N_{\mathcal{L}}(P)] \mid P \in \mathbf{A}(\mathcal{F}) \rangle \in \mathbb{K}$.
- (b) $C_S([\mathcal{L}, \mathcal{L}]) \trianglelefteq \mathcal{L}$.
- (c) *Let Δ^+ be an \mathcal{F} -closed subset of \mathcal{F}^s containing Δ , and let \mathcal{L}^+ be the expansion of \mathcal{L} to Δ^+ given by Theorem II.A1, and let $[\mathcal{L}, \mathcal{L}]^+$ be the expansion of $[\mathcal{L}, \mathcal{L}]$ to a partial normal subgroup of \mathcal{L}^+ given by Theorem II.A2. Then $[\mathcal{L}, \mathcal{L}]^+ = [\mathcal{L}^+, \mathcal{L}^+]$.*

Proof. For brevity, write \mathcal{L}' for $[\mathcal{L}, \mathcal{L}]$, and for any group G write G' for $[G, G]$. By I.1.9 \mathcal{L}' is the union of its subsets \mathcal{L}'_k ($k \geq 0$), where $\mathcal{L}'_0 = \bigcup \{N_{\mathcal{L}}(P) \mid P \in \Delta\}$, and where \mathcal{L}'_{k+1} is the set of all products $\Pi(w)$ with $w \in \mathbf{W}(\mathcal{L}'_k) \cap \mathbf{D}$.

Let Y be the set of all $g \in \mathcal{L}$ for which there exists $x \in \mathcal{L}'$ such that x^g is defined and $x^g \notin \mathcal{L}'$. Assume $Y \neq \emptyset$ and choose $g \in Y$ so that $|S_g|$ as large as possible, and then so that the minimal length of an $\mathbf{A}(\mathcal{L})$ -decomposition for g is as small as possible. Suppose first that $S_g = S$, and let k be the least index for which there exists $x \in \mathcal{L}'_k$ and $g \in N_{\mathcal{L}}(S)$ with $x^g \notin \mathcal{L}'_k$. Then $k > 0$ since, by I.2.3(b), conjugation by g induces an isomorphism $N_{\mathcal{L}}(P) \rightarrow N_{\mathcal{L}}(P^g)$ for any $P \in \Delta$. Thus $x = \Pi(w)$ for some $w \in \mathbf{W}(\mathcal{L}'_{k-1}) \cap \mathbf{D}$. Set $w = (x_1, \dots, x_n)$ and set $w^g = (g^{-1}, x_1, g, \dots, g^{-1}, x_n, g)$. Then $w^g \in \mathbf{D}$ via $(S_w)^g$, and so $x^g = \Pi(w^g) = x_1^g \cdots x_n^g \in \mathcal{L}'_k$. This contradiction (to $g \in Y$) shows that $S_g \neq S$.

Let v be an $\mathbf{A}(\mathcal{L})$ -decomposition for g of minimal length. Then $S_v = S_g$, and hence $v^{-1} \circ (x) \circ v \in \mathbf{D}$ via $S_{(g^{-1}, x, g)}$. The minimality of the length of v then implies that $v = (g)$ is of length 1, and thus $g \in N_{\mathcal{L}}(Q)$ where $Q = S_g \in \mathbf{A}(\mathcal{L})$. Set $H = N_{\mathcal{L}}(Q)$ and $R = N_S(Q)$. Then $g = fh$ for some $f \in H'$ and some $h \in N_H(R)$, and we have $Q = S_{(f, h)}$. Set $u = (h^{-1}, f^{-1}, x, f, h)$. Then $S_u = S_{(g^{-1}, x, g)} \leq Q$, and $x^g = \Pi(u) = (x^f)^h$. Here $x^f \in \mathcal{L}'$ since \mathcal{L}' is a partial group, and then $(x^f)^h \in \mathcal{L}'$ since the maximality of S_g in the choice of g yields $h \notin Y$. Thus $Y = \emptyset$, which is to say that $\mathcal{L}' \trianglelefteq \mathcal{L}$.

The set $\overline{\Delta}$ of objects of the quotient locality $\overline{\mathcal{L}} = \mathcal{L}/\mathcal{L}'$ is the set of all $\overline{P} = P/(S \cap \mathcal{L}')$ such that $S \cap \mathcal{L}' \leq P \in \Delta$. For any such P we have $N_{\overline{\mathcal{L}}}(\overline{P}) \cong N_{\mathcal{L}}(P)/N_{\mathcal{L}'}(P)$, and thus $N_{\overline{\mathcal{L}}}(\overline{P})$ is abelian. Set $\Theta(\overline{P}) = O_{p'}(N_{\overline{\mathcal{L}}}(\overline{P}))$. Then $\Theta(\overline{P})$ centralizes $N_{\overline{S}}(\overline{P})$, and a straightforward argument by induction shows that $\Theta(\overline{P}) \leq \Theta(\overline{S})$ for all $\overline{P} \in \overline{\Delta}$. Thus $\overline{\mathcal{L}} = N_{\overline{\mathcal{L}}}(\overline{S})$ is the group $\overline{S} \times \Theta(\overline{S})$. Since $\overline{S} \trianglelefteq \overline{\mathcal{L}}$, the ‘‘correspondence theorem’’ I.4.5 yields $\mathcal{L}'S \trianglelefteq \mathcal{L}$, and then $\mathcal{L} = \mathcal{L}'N_{\mathcal{L}}(S)$ by the Frattini Lemma (I.3.11). Since $C_S(\mathcal{L}') \trianglelefteq N_{\mathcal{L}}(S)$ and $C_S(\mathcal{L}') \trianglelefteq \mathcal{L}'S$, it follows from I.3.13 that $C_S(\mathcal{L}') \trianglelefteq \mathcal{L}$. Thus (a) holds.

Now let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of $\overline{\mathcal{L}}$ such that \mathcal{L}/\mathcal{N} is an abelian group, and let $\rho : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{N}$ be the canonical projection (see I.4.4). Then ρ maps subgroups of \mathcal{L} to subgroups of \mathcal{L}/\mathcal{N} by I.4.2; hence $N_{\mathcal{L}}(P)' \leq \mathcal{N}$ for all $P \in \Delta$. Thus $\mathcal{L}' \leq \mathcal{N}$, completing the proof of (b).

Let Δ^+ and \mathcal{L}^+ be given as in (c). For any partial normal subgroup $\mathcal{N} \trianglelefteq \mathcal{L}$ let \mathcal{N}^+ be the unique partial normal subgroup of \mathcal{L}^+ which intersects \mathcal{L} in \mathcal{N} , as given by Theorem II.A2. In particular, taking $\mathcal{N} = (\mathcal{L}^+)' \cap \mathcal{L}$, we have $\mathcal{N}^+ = (\mathcal{L}^+)'$. Let $\rho^+ : \mathcal{L}^+ \rightarrow \mathcal{L}^+ / (\mathcal{L}^+)'$ be the canonical projection. Then \mathcal{N} is the kernel of the restriction ρ of ρ^+ to \mathcal{L} . Since $\text{Im}(\rho^+)$ is an abelian group, it follows that $\mathcal{L}' \leq \mathcal{N}$, and so $(\mathcal{L}')^+ \leq (\mathcal{L}^+)'$. On the other hand, we have $\mathcal{L}/\mathcal{L}' \cong \mathcal{L}^+ / (\mathcal{L}')^+$ by II.5.6(a). As \mathcal{L}/\mathcal{L}' is an abelian group we thereby obtain $(\mathcal{L}^+)' \leq (\mathcal{L}')^+$, which completes the proof of (c). \square

Section 4: $\Sigma_T(\mathcal{F})$

The preceding section indicates that the set $\Sigma_T(\mathcal{F})$ introduced in 3.7 plays an important role, if it so happens that $\Sigma_T(\mathcal{F}) \subseteq \Delta$. But the only indication that has been given so far, as to when this condition is met is the one given by lemma 3.15: in which there is the rather strong assumption that $O_p(\mathcal{L})\mathcal{E}^c \subseteq \Delta$. One aim of this section is to show that one indeed has $\Sigma_T(\mathcal{F}) \subseteq \Delta$ provided only that Δ is large enough. For example, it will suffice that Δ be as large as possible, i.e. that Δ be the set \mathcal{F}^s of \mathcal{F} -subcentric

subgroups of S .

We continue the setup of 2.1. Thus (\mathcal{L}, Δ, S) is a proper locality on \mathcal{F} , $\mathcal{N} \trianglelefteq \mathcal{L}$ is a fixed partial normal subgroup of \mathcal{L} , $T = S \cap \mathcal{N}$, and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$. We also continue the notation: $H = N_{\mathcal{L}}(C_S(T)T)$ (a subgroup of \mathcal{L} by 2.2), $\mathcal{L}_T = N_{\mathcal{L}}(T)$, and $\mathcal{C}_T = C_{\mathcal{L}}(T)$. Recall from 1.5 that $(\mathcal{L}_T, \Delta, S)$ is a proper locality on $N_{\mathcal{F}}(T)$, and that $\mathcal{C}_T \trianglelefteq \mathcal{L}_T$.

Lemma 4.1. *Assume that $S = C_S(T)T$, and assume that $\mathcal{F}^c \subseteq \Delta$.*

- (a) $UV \in \mathcal{F}^c$ for all $(U, V) \in \mathcal{E}^c \times C_{\mathcal{F}}(T)^c$.
- (b) *Let V be a subgroup of $C_S(T)$ containing $Z(T)$, and with $TV \in \mathcal{F}^c$. Then $V \in C_{\mathcal{F}}(T)^c$.*

Proof. First, let $U \leq T$ be a subgroup of T , and let $U' \in U^{\mathcal{F}}$. Thus there exists $w = (g_1, \dots, g_n) \in \mathbf{W}(\mathcal{L})$, and a sequence (U_0, \dots, U_n) of \mathcal{F} -conjugates of U such that $U_0 = U$, $U_n = U'$, and $U_k = (U_{k-1})^{g_k}$ for all k with $1 \leq k \leq n$. Write $g_k = x_k y_k$ where $x_k \in \mathcal{N}$ and $y_k \in C_{\mathcal{L}}(T)$. We have $N_{\mathcal{N}}(T) \leq N_{\mathcal{L}}(C_S(T)T)$ by I.3.5, and $S_{(x_k, y_k)} = S_{g_k}$ by I.3.10 and I.3.12. It follows that $U_k = (U_{k-1})^{x_k}$, and thus:

- (1) $U^{\mathcal{F}} = U^{\mathcal{E}}$.

Assume now that (a) is false, and among all $(U, V) \in \mathcal{E}^c \times C_{\mathcal{F}}(T)^c$ with $UV \notin \mathcal{F}^c$ choose (U, V) so that $|U||V|$ is as large as possible. Set $R = UV$. Notice that $N_T(U) \leq N_S(R)$ and that $N_{C_S(T)}(V) \leq N_S(R)$. As $S = C_S(T)T$ we have $N_S(R) = N_T(U)N_{C_S(T)}(V)$, so the maximality of $|U||V|$ yields $N_S(R) \in \mathcal{F}^c$.

By II.1.15 there exists an \mathcal{F} -conjugate R' of R such that both R' and $R' \cap T$ are fully normalized in \mathcal{F} . There is then an \mathcal{F} -homomorphism $\lambda : N_S(R) \rightarrow N_S(R')$ such that $R' = R\lambda$. As $N_S(R) \in \Delta$, λ is given by conjugation by some $g \in \mathcal{L}$. We may employ I.3.12 in order to write $g = fy$, with $f \in N_{\mathcal{L}}(T)$, $y \in \mathcal{N}$, and with $S_g = S_{(f, y)}$. Set $U_1 = U^f$ and $V_1 = V^f$. Then $U_1 \in U^{\mathcal{E}}$ by (1), and so $U_1 \in \mathcal{E}^c$. Applying I.3.12 also to \mathcal{L}_T and its partial normal subgroup \mathcal{C}_T , we obtain $f = hx$ where $h \in H$, $x \in \mathcal{C}_T$, and $S_f = S_{(h, x)}$. Here $H = N_{\mathcal{L}}(S)$ since $S = C_S(T)T$, so conjugation by h is defined on all of \mathcal{L} , and then 2.3 shows that conjugation by h preserves $C_{\mathcal{F}}(T)^c$. Thus $V_1 \in C_{\mathcal{F}}(T)^c$, and we may therefore assume that $U = U_1$ and $V = V_1$, and that $g = y$. Then I.3.1(b) yields:

- (2) $V^g \leq VT$.

Set $U' = U^y$ and $V' = V^y$. We now compute:

$$\begin{aligned}
C_S(R') &= C_S(U'V') = C_S(V') \cap C_S(U') = C_S(V') \cap C_{C_S(T)T}(U') \\
&= C_S(V') \cap C_S(T)Z(U') \quad (\text{as } U' \text{ is centric in } \mathcal{E}) \\
&= C_{C_S(T)}(V')Z(U') \quad (\text{as } [U', V'] \cong [U, V] = 1). \\
&= C_{C_S(T)}(V'T)Z(U') = C_{C_S(T)}(VT)Z(U') \quad (\text{by (2)}) \\
&= Z(V)Z(U').
\end{aligned}$$

Thus, in order to show that R' , and hence R , is centric in \mathcal{F} it suffices to show that $Z(V) \leq R'$. Since $C_S(R) = Z(V)Z(U)$, it then suffices to show that $|Z(V)Z(U')| =$

$|Z(V)Z(U)|$. As $V \leq C_S(T)$, and since $U, U' \in \mathcal{E}^c$ we have:

$$Z(V) \cap Z(U') = Z(V) \cap U' = Z(V) \cap T = Z(V) \cap U = Z(V) \cap Z(U).$$

Thus $|Z(V) \cap Z(U')| = |Z(V) \cap Z(U)|$, and hence $|Z(V)Z(U')| = |Z(V)Z(U)|$, as required. Thus $R \in \mathcal{F}^c$, and this contradiction completes the proof of (a).

Now let V be a subgroup of $C_S(T)$ containing $Z(T)$, and with $TV \in \mathcal{F}^c$, and suppose that $V \notin C_{\mathcal{F}}(T)^c$. Let V' be a $C_{\mathcal{F}}(T)$ -conjugate of V , with V' fully centralized in $C_{\mathcal{F}}(T)$. Then $C_{C_S(T)}(V') \not\leq V'$ by II.1.10. That is, we have $C_S(TV') \not\leq V'$. On the other hand, since $TV' \cap T = T$ and $TV' \cap C_S(T) = Z(T)V' = V'$, we have

$$C_S(TV') \cap TV' = Z(TV') = Z(T)Z(V') \leq V'.$$

This shows that $C_S(TV') \not\leq TV'$, and so $TV' \notin \mathcal{F}^c$. But $C_{\mathcal{F}}(T) = \mathcal{F}_{C_S(T)}(\mathcal{C}_T)$ by 1.5(b), and thus TV' is an \mathcal{F} -conjugate of TV via the same sequence of conjugation maps by elements of \mathcal{C}_T that sends V to V' . As $TV \in \mathcal{F}^c$ and $TV' \notin \mathcal{F}^c$, we have the contradiction which proves (b). \square

Corollary 4.2. *If $S = C_S(T)T$ and $\mathcal{F}^c \subseteq \Delta$, then $\Sigma_T(\mathcal{F}) \subseteq \Delta$.*

Proof. Let $X = O_p(\mathcal{L})UV \in \Sigma_T(\mathcal{F})$, with U and V given as in definition 3.7. Then $U \in \mathcal{E}^c$ by 2.6(b), while $V \in C_{\mathcal{F}}(T)^c$ by definition. The preceding lemma then yields $UV \in \mathcal{F}^c$, and hence $X \in \Delta$. \square

Lemma 4.3. *Assume $\Sigma_T(\mathcal{F}) \subseteq \Delta$ and assume $T \in \mathcal{F}^q$.*

- (a) *If $\Delta = \mathcal{F}^s$ then $(\mathcal{N}, \mathcal{E}^c)$ is a localizable pair, and $(\mathcal{N}_{\mathcal{E}^c}, \mathcal{E}^c, T)$ is a proper locality on \mathcal{E} .*
- (b) *If $\Delta = \mathcal{F}^s$ then the image of the natural homomorphism $\lambda : \mathcal{L}_T \rightarrow \text{Aut}(T)$ is contained in $\text{Aut}(\mathcal{E})$.*
- (c) *$\mathcal{E}^c \subseteq \mathcal{F}^q$, and $\mathcal{E}^s \subseteq \mathcal{F}^s$.*
- (d) *\mathcal{E} is (cr)-generated.*

Proof. If points (c) and (d) hold in the case that $\Delta = \mathcal{F}^s$ then they hold in general, by 3.14. Thus, we may assume throughout that $\Delta = \mathcal{F}^s$.

Set $\mathcal{U}^c = \{U \in \mathcal{E}^c \mid U^{\mathcal{F}} \subseteq \mathcal{E}^c\}$, and set $\mathcal{U}^s = \{U \in \mathcal{E}^s \mid U^{\mathcal{F}} \subseteq \mathcal{E}^s\}$. The strategy of the proof is to first establish all three parts of the lemma under the following assumption.

- (1) $\mathcal{U}^c = \mathcal{E}^c \subseteq \Delta$.

Assume (1). Then 3.15 yields (a) and (b), and (d) then follows from (a) and from II.2.10. Further, (b) implies that $\mathcal{U}^s = \mathcal{E}^s$. Pick $U \in \mathcal{E}^s$. In order to show that $U \in \mathcal{F}^s$ we may assume that U is fully normalized in \mathcal{F} . Set $X = O_p(N_{\mathcal{E}}(U))$. Then $X \in \mathcal{E}^c$, and so $X \in \mathcal{F}^s$ (and $X \in \Delta$) by (1). Further, II.1.16 implies that X is fully centralized in \mathcal{F} . Set $M = N_{\mathcal{L}}(X)$ and $K = C_{\mathcal{L}}(X)$. Then K is of characteristic p , and $X = O_p(K)$ by II.2.3.

Set $Y = O_p(N_{\mathcal{F}}(U))$. Then $X \trianglelefteq Y$, and then $Y = O_p(N_M(U))$. As $X \leq Y$ we may compute:

$$C_{\mathcal{L}}(Y) = C_M(Y) = C_{N_M(U)}(Y).$$

As $N_M(U)$ is of characteristic p by II.2.6(b), we conclude that $C_{\mathcal{L}}(Y) \leq Y$. Then $Y \in \mathcal{F}^c$ by II.2.8(b), and $U \in \mathcal{F}^s$. Thus $\mathcal{E}^s \subseteq \mathcal{F}^s$.

It now suffices to prove the following stronger version of (1).

- (2a) $\mathcal{U}^c \subseteq \mathcal{F}^q$, and
- (2b) $\mathcal{U}^c = \mathcal{E}^c$.

Assume that (2a) is false, and choose $U \in \mathcal{U}^c - \mathcal{F}^q$ with $|U|$ as large as possible. As $U \in \mathcal{U}^c$ we may assume that U is fully normalized in \mathcal{F} . Set $\Gamma = \{X \in \Delta \mid U \trianglelefteq X\}$ and $\Sigma = \{Y \leq C_S(U) \mid UY \in \Delta\}$. By 1.4 (and since $\mathcal{F}^c \subseteq \Delta$) both $(\mathcal{N}_{\mathcal{L}}(U), \Gamma)$ and $(C_{\mathcal{L}}(U), \Sigma)$ are localizable pairs. Write \mathcal{L}_U for $N_{\mathcal{L}}(U)_{\Gamma}$, and \mathcal{C}_U for $C_{\mathcal{L}}(U)_{\Sigma}$. Then, by 1.4, $(\mathcal{L}_U, \Gamma, N_S(U))$ is a proper locality on $N_{\mathcal{F}}(U)$, $(\mathcal{C}_U, \Sigma, C_S(U))$ is a proper locality on $C_{\mathcal{F}}(U)$, and Evidently $\mathcal{C}_U = C_{\mathcal{L}_U}(U)$, and so $\mathcal{C}_U \trianglelefteq \mathcal{L}_U$.

Set $B = N_T(U)$. Thus U is properly contained in B , and so $B \in \mathcal{F}^q$. Let $V \in C_{\mathcal{F}}(U)^{cr}$ with V fully normalized in $C_{\mathcal{F}}(U)$, and set $D = N_{\mathcal{C}_U}(V)$. Then $V \in \Delta$ by 1.4(a), so D is a subgroup of \mathcal{C}_U , and $D = N_{\mathcal{C}_U}(UV)$. Set $K = \mathcal{N} \cap D$. Then $B \leq K \trianglelefteq D$, and then

$$(*) \quad [B, D] \leq C_K(U) = Z(U) \times O_{p'}(K),$$

since $B \in \mathcal{E}^c$. As \mathcal{L}_U is a proper locality, $N_{\mathcal{L}_U}(UV)$ is of characteristic p , so also D and K are of characteristic p by II.2.6. Thus $O_{p'}(K) = 1$, and so $(*)$ yields $[B, D] \leq Z(U)$. Then $[B, O^p(D)] = 1$ by II.2.6(c). As $B \in \mathcal{F}^q$ we conclude that $O^p(D) = 1$. As $V \in C_{\mathcal{F}}(V)^{cr}$ it follows that $V = C_S(U) = \mathcal{C}_U$. Then $C_{\mathcal{F}}(U)$ is the trivial fusion system on $C_S(U)$, and hence $U \in \mathcal{F}^q$. This completes the proof of (2a).

Finally, assume (2b) to be false, and among all $U \in \mathcal{E}^c - \mathcal{U}^c$, choose U so that $|U|$ is as large as possible. Then $U \neq T$, so U is a proper subgroup of $N_T(U)$, and then $N_T(U) \in \mathcal{F}^q$ by (2a). Thus $N_T(U) \in \Delta$, and we may then argue - in a perhaps familiar way - as follows. Let U' be a fully normalized \mathcal{F} -conjugate of U , and let $\phi : N_S(U) \rightarrow S$ be an \mathcal{F} -homomorphism such that $U\phi = U'$. Then ϕ is given by conjugation by an element $g \in \mathcal{L}$ (as $N_T(U) \in \Delta$), and $g = xh$ for some $x \in \mathcal{N}$ and $h \in \mathcal{L}_T$ with $S_g = S_{(x,h)}$ (I.2.12). As \mathcal{E}^c is both \mathcal{E} -invariant and \mathcal{L}_T -invariant, we then have $U' \in \mathcal{E}^c$, and a similar argument then show that $(U')^{\mathcal{F}} \subseteq \mathcal{E}^c$. Thus $U \in \mathcal{U}^c$ after all. This completes the proof of (2b), and thereby proves the lemma. \square

Remark. It appears that point (c) of the preceding lemma cannot be improved upon in any obvious way. For example, \mathcal{E}^{cr} need not be contained in \mathcal{F}^c , even if $T \in \mathcal{F}^c$. For example, let G be a semi-direct product $V \rtimes H$ where $H = GL_3(2)$ and where V is elementary abelian of order 16, with $C_V(H) = 1$. Then G may be viewed as a proper locality whose objects are the overgroups of V in a Sylow 2-subgroup S of G . Let N be the subgroup of G of index 2 in G , and set $T = S \cap N$. Then T is centric in $\mathcal{F}_S(G)$, and $V \cap T$ is centric radical in $\mathcal{F}_T(\mathcal{N})$; but $V \cap T$ is not centric in $\mathcal{F}_S(G)$.

Our aim now is to use 4.1 and 4.3 to show that $\Sigma_T(\mathcal{F}) \subseteq \mathcal{F}^q$ if $\Delta = \mathcal{F}^s$. At the same time, we want to obtain information about the special case where $C_S(\mathcal{N})$ is contained in \mathcal{N} . The next result (a corollary of 3.11) prepares the way for these goals.

Lemma 4.4. Assume $\Sigma_T(\mathcal{F}) \subseteq \Delta$, and let $\tilde{\Sigma}_T(\mathcal{F})$ be the set of all $X \in \Sigma_T(\mathcal{F})$ such that $C_S(T) \leq X$. Then

$$(*) \quad C_S(\mathcal{N}) = \bigcap \{C_S(N_{\mathcal{N}}(X)) \mid X \in \tilde{\Sigma}_T(\mathcal{F})\}.$$

Proof. Let R be the right-hand intersection in (*). Then $C_S(\mathcal{N}) \leq R$, so it remains to prove the reverse inclusion. Let $f \in \mathcal{N}$ and let $g \in R$. As $\Sigma_T(\mathcal{F}) \subseteq \Delta$ by hypothesis, there exists an \mathcal{N} -decomposition $w = (a, f_1, \dots, f_n)$ of f , and $a \in \mathcal{N} \cap H$, by 3.12. Set $w' = (g^{-1}, a, g, g^{-1}, f_1, g, \dots, g^{-1}, f_n, g)$. Then $w' \in \mathbf{D}$ via $(S_w)^g$. By definition 3.11, each f_i normalizes a member of $\tilde{\Sigma}_T(\mathcal{F})$, so $(f_i)^g = f_i$. As $C_S(T)T \in \tilde{\Sigma}_T(\mathcal{F})$, also $a^g = a$. Thus $f^g = \Pi(w') = \Pi(w) = g$, and so $g \in C_S(\mathcal{N})$. Thus $R \leq C_S(\mathcal{N})$, and the reverse inclusion is obvious. \square

Proposition 4.5. Assume $\Sigma_T(\mathcal{F}) \subseteq \Delta$, and assume that $C_S(\mathcal{N}) \leq \mathcal{N}$. Set $\Gamma = \mathcal{E}^s$.

- (a) $\mathcal{E}^c \subseteq \mathcal{F}^q$, and $\Sigma_T(\mathcal{F}) \subseteq \mathcal{F}^q$.
- (b) If $\Delta = \mathcal{F}^s$, then $(\mathcal{N}, \mathcal{E}^s)$ is a localizable pair, and $(\mathcal{N}_\Gamma, \Gamma, T)$ is a proper locality on \mathcal{E} .
- (c) If $\Delta = \mathcal{E}^s$ then \mathcal{L}_T is a subgroup of \mathcal{L} , and the image of \mathcal{L}_T in $\text{Aut}(T)$ is a subgroup of $\text{Aut}(\mathcal{N}_\Gamma)$.
- (d) If $\Delta = \mathcal{F}^s$ then $O_p(\mathcal{E}) = O_p(\mathcal{N}_\Gamma) = O_p(\mathcal{N}) \trianglelefteq \mathcal{L}$.

Proof. It follows from 3.14 that if (a) holds if it holds under the assumption that $\Delta = \mathcal{F}^s$. We may therefore assume throughout that $\Delta = \mathcal{F}^s$.

The proof will have two parts. In the first we assume:

$$(*) \quad \Sigma_T(\mathcal{F}) \subseteq \mathcal{F}^q.$$

Given (*), we will show that (a) through (d) hold. Once that has been achieved, we will then be able to show that the hypothesis (*) is redundant, and to thereby complete the proof.

Assume (*). Define $w\Sigma_T(\mathcal{F})$ as in 4.4, let $X \in \tilde{\Sigma}_T(\mathcal{F})$, and set $U = X \cap T$. Then $X = O_p(\mathcal{L})C_S(T)U$. Let $V \in C_{\mathcal{F}}(T)^{cr}$ with V fully normalized in $C_{\mathcal{F}}(T)$. Then $O_p(\mathcal{L})TV \in \Sigma_T(\mathcal{F})$, so $O_p(\mathcal{L})TV \in \Delta$. Then $N_{C_{\mathcal{F}}(T)}(V)$ is the fusion system of $N_{C_T}(TV)$ over $C_{C_S(T)}(V)$, by II.2.2. As \mathcal{L} is proper, $N_{\mathcal{L}}(TV)$ is of characteristic p . Since $N_{\mathcal{L}}(TV) = N_{\mathcal{L}_T}(V)$, and since $\mathcal{C}_T \trianglelefteq \mathcal{L}_T$, the group $K := O^p(N_{\mathcal{C}_T}(V))$ is of characteristic p by II.2.6(a). Set $V_0 = V \cap K$. Then $V_0 \leq C_S(\mathcal{N})$, by 3.9 and 4.4. As $C_S(\mathcal{N}) \leq T$ by hypothesis, we then have $V_0 \leq Z(K)$. Then $[V, K] = 1$ by II.2.6(c), and then $K = 1$ since V is centric in $C_{\mathcal{F}}(T)$. As V is also radical in $C_{\mathcal{F}}(T)$ we conclude that $V = C_S(T)$. Thus $C_{\mathcal{F}}(T)$ is the trivial fusion system on $C_S(T)$. That is, we have $T \in \mathcal{F}^q$. Point (a) is then given by 4.3(c) and by (*). Points (b) and (c) are given by the relevant points in 4.3, in conjunction with expansion from \mathcal{E}^c to \mathcal{E}^s via Theorem II.A.

Next, it follows from (b) and II.2.3 that $O_p(\mathcal{E}) = O_p(\mathcal{N}_\Gamma)$, and from (c) that $O_p(\mathcal{E})$ is H -invariant. As $\mathcal{N}_\Gamma = \mathcal{N} \cap \mathcal{L}_0$ we have $\mathcal{N}_\Gamma \trianglelefteq \mathcal{L}_0$. As $\mathcal{L}_0 = \mathcal{N}\mathcal{L}_T$ we obtain $O_p(\mathcal{E}) \trianglelefteq \mathcal{L}_0$, and then $O_p(\mathcal{E}) \trianglelefteq \mathcal{L}$ by Theorem II.A2. This establishes (d). It now remains to remove the hypothesis (*).

Let \mathcal{M} be the partial subgroup $\langle \mathcal{C}_T \mathcal{N} \rangle$ of \mathcal{L} . Then $\mathcal{M} \trianglelefteq \mathcal{L}$ and $S \cap \mathcal{M} = C_S(T)T$, by I.5.5. Set $\tilde{T} = C_S(T)T$ and set $\mathcal{D} = \mathcal{F}_{\tilde{T}}(\mathcal{M})$. Then $\tilde{T} \in \mathcal{F}^c$, so 4.3(a) implies that $(\mathcal{M}, \mathcal{D}^c)$ is a localizable pair and that $\mathcal{M}_{\mathcal{D}^c}$ is a proper locality on \mathcal{D} . Then $\Sigma_T(\mathcal{F}) \subseteq \mathcal{D}^c$ by 4.1(a), with \mathcal{M} in the role of \mathcal{L} . Since $\mathcal{D}^c \subseteq \mathcal{F}^q$ by 4.3(c) we thereby obtain $\Sigma_T(\mathcal{L}) \subseteq \mathcal{F}^q$. Thus (*) holds, and the proof is complete. \square

Corollary 4.6. *Assume the setup of 2.1. Then $\Sigma_T(\mathcal{F}) \subseteq \mathcal{F}^q$.*

Proof. By 3.8 the definition of $\Sigma_T(\mathcal{F})$ depends only on \mathcal{F} and on the strongly closed subgroup T , and not on Δ . By Theorem II.A we may therefore assume that $\Delta = \mathcal{F}^s$. In this case the proof is given by repeating - verbatim - the final paragraph in the proof of 4.5. \square

Corollary 4.7. *Assume the setup of 2.1, and assume that T is abelian. Then $T \trianglelefteq \mathcal{N}$, and \mathcal{N} is a subgroup of $N_{\mathcal{L}}(C_S(T)T)$. Moreover, if $T \leq Z(\mathcal{N})$ then $\mathcal{N} = T$.*

Proof. As $\Sigma_T(\mathcal{F})\mathcal{F}^q \subseteq \mathcal{F}^s$ there exists an \mathcal{F} -closed set Δ^+ of subgroups of S such that $\Delta \cup \Sigma_T(\mathcal{F}) \subseteq \Delta^+ \subseteq \mathcal{F}^s$. Let \mathcal{L}^+ be the expansion of \mathcal{L} to Δ^+ via Theorem II.A1, and let \mathcal{N}^+ be the partial normal subgroup of \mathcal{L}^+ corresponding to \mathcal{N} via Theorem II.A2. Let $g \in \mathcal{N}$. Then $g \in \mathcal{N}^+$, so g has an \mathcal{N}^+ -decomposition $w = (a, g_1, \dots, g_n)$. For each index i (if any) with $1 \leq i \leq n$ there exists $X_i \in \Sigma_T(\mathcal{F})$ such that $T \not\leq X_i$, $N_T(X_i) \in \text{Syl}_p(N_{\mathcal{N}}(X_i))$, and $g_i \in O^p(N_{\mathcal{N}}(X_i))$. As $N_T(X_i)$ is abelian, these conditions imply that $O^p(N_{\mathcal{N}}(X_i)) = 1$, and so $w = (a)$. Thus $g \in N_{\mathcal{N}}(C_S(T)T)$ by 3.12.

Now \mathcal{N} is a subgroup of the group $H := N_{\mathcal{L}}(C_S(T)T)$. Assume that $T \leq Z(\mathcal{N})$. Then $\mathcal{N} = T \times O_{p'}(\mathcal{N})$, and then $O_{p'}(\mathcal{N}) = 1$ since H is of characteristic p . Thus $\mathcal{N} = T$ in this case. \square

Recall (cf. 2.7) that $Z(\mathcal{N})$ is the set of all $z \in \mathcal{N}$ such that g^z is defined and is equal to f for all $g \in \mathcal{N}$.

Lemma 4.8. *Suppose $\Sigma_T(\mathcal{F}) \subseteq \Delta$. Then $C_S(\mathcal{N}) \trianglelefteq H$, and $Z(\mathcal{N}) = C_T(\mathcal{N}) \trianglelefteq \mathcal{L}$.*

Proof. Define $\tilde{\Sigma}_T(\mathcal{F})$ as in 4.4. Thus, each $X \in \tilde{\Sigma}$ is of the form $O_p(\mathcal{L})C_S(T)U$ where $U = (P \cap T)^h$ for some $P \in \mathbf{A}_T(\mathcal{F})$ and some $h \in H$. As $C_S(T) \trianglelefteq H$, $\tilde{\Sigma}_T(\mathcal{F})$ is H -invariant. Since 4.4(*) can be expressed as

$$C_S(\mathcal{N}) = C_S(T) \cap \left(\bigcap \{C_S(N_{\mathcal{N}}(X)) \mid X \in \tilde{\Sigma}_T(\mathcal{F})\} \right),$$

it follows that $C_S(\mathcal{N}) \trianglelefteq H$. As $Z(\mathcal{N}) = C_T(\mathcal{N})$ by 2.7, we obtain

$$Z(\mathcal{N}) = T \cap C_S(\mathcal{N}) \trianglelefteq H.$$

As $C_T(\mathcal{N}) \trianglelefteq \mathcal{C}_T$, we may apply I.3.13 with \mathcal{L}_T and \mathcal{C}_T in the roles of \mathcal{L} and \mathcal{N} to obtain $Z(\mathcal{N}) \trianglelefteq \mathcal{L}_T$. Then apply I.3.13 to \mathcal{L} and \mathcal{N} to obtain $Z(\mathcal{N}) \trianglelefteq \mathcal{L}$. \square

Section 5. \mathcal{N}^\perp

We continue the setup and the notation of the preceding sections. Thus (\mathcal{L}, Δ, S) is a proper locality on \mathcal{F} , \mathcal{N} is a partial normal subgroup of \mathcal{L} , $T = S \cap \mathcal{N}$, and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$. Further, we have the abbreviations $\mathcal{L}_T = N_{\mathcal{L}}(T)$, $\mathcal{C}_T = C_{\mathcal{L}}(T)$, and $H = N_{\mathcal{L}}(C_S(T)T)$. The collection $\Sigma_T(\mathcal{F})$ of subgroups of S , defined in 3.7, will continue to play a key role.

In this section we will be taking a roundabout path to the structure of \mathcal{L} and \mathcal{N} by way of the structure of \mathcal{L}_T and \mathcal{C}_T . Recall from 2.2 that H is a subgroup of \mathcal{L} , and from 1.5 that $(\mathcal{L}_T, \Delta, S)$ is a proper locality on $N_{\mathcal{F}}(T)$, that $\mathcal{C}_T \trianglelefteq \mathcal{L}_T$, and that $C_{\mathcal{F}}(T) = \mathcal{F}_{C_S(T)}(\mathcal{C}_T) \subseteq \Delta$.

Theorem 5.1. *Assume $O_p(\mathcal{L})\mathcal{E}^{cr} \subseteq \Delta$, and assume that \mathcal{E} is (cr)-generated. Let λ be a mapping which assigns to each $U \in \mathcal{E}^{cr}$ a group $\lambda(U)$, with*

$$\lambda(U) \trianglelefteq N_{\mathcal{L}}(U) \quad \text{and} \quad \lambda(U) \leq N_{\mathcal{N}}(U).$$

Set $\mathcal{M}(\lambda) = \langle \lambda(U) \mid U \in \mathcal{E}^{cr} \rangle$. Then $\lambda(U)^h$ and $\lambda(U^h)$ are defined for all $h \in \mathcal{L}_T$, and $\mathcal{M}(\lambda) \trianglelefteq \mathcal{L}$ if the following two conditions hold for all $U \in \mathcal{E}^{cr}$.

- (1) $O^p(N_{\mathcal{N}}(U)) \cap O^{p'}(N_{\mathcal{N}}(U)) \leq \lambda(U)$, and
- (2) $\lambda(U)^h = \lambda(U^h)$ for all $h \in \mathcal{L}_T$.

Proof. Let Γ be the overgroup closure of \mathcal{E}^{cr} in T . Then $O_p(\mathcal{L})\Gamma \subseteq \Delta$, and by 2.8 shows that (\mathcal{N}, Γ) is a localizable pair, that $(\mathcal{N}_{\Gamma}, \Gamma, T)$ is a proper locality on \mathcal{E} , \mathcal{L}_T is a subgroup of \mathcal{L} which acts on \mathcal{N}_{Γ} by conjugation, and the image of the natural homomorphism $\mathcal{L}_T \rightarrow \text{Aut}(T)$ is contained in $\text{Aut}(\mathcal{E})$. In particular, $\lambda(U)^h$ and $\lambda(U^h)$ are defined for $U \in \mathcal{E}^{cr}$ and $h \in \mathcal{L}_T$.

Let Δ_0 be the overgroup closure of $O_p(\mathcal{L})\Gamma$ in S . Then $\mathcal{F}^{cr} \subseteq \Delta_0 \subseteq \Delta$ by 2.6(b), the restriction $(\mathcal{L}_0, \Delta_0, S)$ of \mathcal{L} to Δ_0 is a proper locality on \mathcal{F} , and $\mathcal{N}_{\Gamma} = \mathcal{N} \cap \mathcal{L}_0$ by 2.8(a). We shall write \mathcal{N}_0 for \mathcal{N}_{Γ} . Let \mathcal{M}_0 be the partial subgroup of \mathcal{L}_0 generated by $\{\lambda(V) \mid V \in \mathcal{E}^{cr}\}$.

Set $\mathcal{U} = \{(P \cap T)^h \mid P \in \mathbf{A}_T(\mathcal{L}), h \in N_{\mathcal{L}}(C_S(T)T)\}$. Then $\mathcal{U} \subseteq \mathcal{E}^{cr}$ by 2.6(b), and so $\Sigma_T(\mathcal{F}) \subseteq \Delta_0$. Then every element of \mathcal{L}_0 has an \mathcal{N}_0 -decomposition, by 3.12. Let $f \in \mathcal{M}_0$ and let $g \in \mathcal{L}_0$ such that f^g is defined in \mathcal{L}_0 . Let $u = (a, f_1, \dots, f_m)$ be an \mathcal{N} -decomposition of f , let $v = (b, g_1, \dots, g_n)$ be an \mathcal{N} -decomposition of g , and set $w = v^{-1} \circ u \circ v$. Thus

$$w = (g_n^{-1}, \dots, g_1^{-1}, b^{-1}, a, f_1, \dots, f_m, b, g_1, \dots, g_n),$$

and $S_w = S_{(g^{-1}, f, g)} \in \Delta_0$. By (1) and definition 3.11, there exist elements U_i and V_j of \mathcal{U} (and hence of \mathcal{E}^{cr}) such that $f_i \in \lambda(U_i)$ and $g_j \in \lambda(V_j)$. Further, we have $a, b \in N_{\mathcal{L}}(T)$. As $f \in \mathcal{M}_0$, and $f = a(f_1 \cdots f_m)$ with $f_1 \cdots f_m \in \mathcal{M}_0$, the Dedekind lemma (I.1.10) implies that $a \in \mathcal{M}_0$, and so $a \in N_{\mathcal{M}_0}(T)$. We may employ the Frattini Calculus (I.3.4), to obtain a word

$$(*) \quad w' = (b^{-1}, a, b, g'_n, \dots, g'_1, f'_1, \dots, f'_m, g_1, \dots, g_n)$$

such that $S_w = S_{w'}$, $\Pi(w) = \Pi(w')$, and (by (2)) with g'_j and f'_i in \mathcal{M}_0 for all i and j .

Set $Y_0 = \bigcup \{\lambda(V) \mid V \in \mathcal{E}^{cr}, \text{ and for each } k > 0 \text{ set}$

$$Y_k = \{\Pi(\tilde{u}) \mid \tilde{u} \in \mathbf{W}(Y_{k-1}) \cap \mathbf{D}(\mathcal{L}_0)\}.$$

Then \mathcal{M}_0 is (by I.1.9) the union of the sets Y_k . We now show by induction on k that each Y_k is invariant under conjugation by \mathcal{L}_T . By (2), Y_0 is \mathcal{L}_T -invariant. Suppose that Y_{k-1} is \mathcal{L}_T -invariant, let $\tilde{u} = (y_1, \dots, y_d) \in Y_k$, and let $h \in \mathcal{L}_T$. Then $S_{\tilde{u}} \cap O_p(\mathcal{L})T \in \Delta_0$, so the word

$$\tilde{u}^h = (h^{-1}, y_1, h, \dots, h^{-1}, y_d, h)$$

is in $\mathbf{D}(\mathcal{L}_0)$. Since each of $(y_1)^h, \dots, (y_d)^h$ is in Y_{k-1} , and since $\Pi(\tilde{u}^h) = \Pi(\tilde{u})^h$ (by $\mathbf{D}(\mathcal{L}_0)$ -associativity), the induction is complete. Thus \mathcal{L}_T acts on \mathcal{M}_0 by conjugation. In particular, $N_{\mathcal{M}_0}(T)$ is a normal subgroup of \mathcal{L}_T , and so $a^b \in \mathcal{M}_0$. Now (*) yields $f^g = a^b f'$ for some $f' \in \mathcal{M}_0$, and so $f^g \in \mathcal{M}_0$. Thus $\mathcal{M}_0 \trianglelefteq \mathcal{L}_0$.

Set $\mathcal{M} = \mathcal{M}(\lambda)$, and let \mathcal{M}^+ be the partial subgroup of \mathcal{L} generated by the set $\mathcal{M}^{\mathcal{L}}$ of all \mathcal{L} -conjugates of elements of \mathcal{M} . Thus:

$$\mathcal{M}_0 \subseteq \mathcal{M} \leq \mathcal{M}^+.$$

Since Δ_0 is contained in Δ , we have $\Sigma_T(\mathcal{L}) \subseteq \Delta$, and so each element of \mathcal{L} has an \mathcal{N} -decomposition. Let f now be an arbitrary element of \mathcal{M}^+ , and let $u = (a, f_1, \dots, f_m)$ be an \mathcal{N} -decomposition (in \mathcal{L}) of f . As we have already seen, (1) yields $f_i \in \mathcal{M}_0$ for all i . Then $f_1 \cdots f_m \in \mathcal{M}$, and $a \in N_{\mathcal{M}^+}(T)$. By Theorem II.A2, we have $\mathcal{M}^+ \trianglelefteq \mathcal{L}$, and $\mathcal{L}_0 \cap \mathcal{M}^+ = \mathcal{M}_0$. Then

$$N_{\mathcal{M}^+}(T) = N_{\mathcal{L}}(T) \cap \mathcal{M}^+ = N_{\mathcal{L}}(T) \cap \mathcal{M}_0,$$

as $N_{\mathcal{L}}(T) \leq \mathcal{L}_0$. This shows that $a \in \mathcal{M}_0$, and that $f = \Pi(u) \in \mathcal{M}$. Thus $\mathcal{M} = \mathcal{M}^+$, and so $\mathcal{M} \trianglelefteq \mathcal{L}$. \square

Remark. The hypotheses of the preceding proposition are fulfilled trivially if $\mathcal{N} = \mathcal{L}$, and by the mapping $U \mapsto O^p(N_{\mathcal{N}}(U))$ (and similarly for $U \mapsto O^{p'}(N_{\mathcal{N}}(U))$).

By II.7.2 there is a smallest partial normal subgroup $\mathcal{K} := O_{\mathcal{L}}^p(\mathcal{N})$ of \mathcal{L} such that $\mathcal{K}T = \mathcal{N}$, and a smallest partial normal subgroup $\mathcal{K}' = O_{\mathcal{L}}^{p'}(\mathcal{N})$ of \mathcal{L} such that $T \leq \mathcal{K}$. In the following arguments \mathcal{K} will play a decisive role, and it will be useful to know that the defining properties of \mathcal{K} are invariant under change of objects. Thus, by II.7.3, if $\Delta \subseteq \Delta^+ \subseteq \mathcal{F}^s$ and Δ^+ is \mathcal{F} -closed, then

$$O_{\mathcal{L}^+}^p(\mathcal{N}^+) = O_{\mathcal{L}}^p(\mathcal{N})^+$$

(and the analogous property holds for $O_{\mathcal{L}}^{p'}(\mathcal{N})$).

Corollary 5.2. Assume $O_p(\mathcal{L})\mathcal{E}^{cr} \subseteq \Delta$, and define mappings λ and λ' on \mathcal{E}^{cr} by

$$\lambda(U) = O^p(N_{\mathcal{N}}(U)) \quad \text{and} \quad \lambda'(U) = O^{p'}(N_{\mathcal{N}}(U)).$$

Then $O_{\mathcal{L}}^p(\mathcal{N}) = \mathcal{M}(\lambda)$, and $O_{\mathcal{L}}^{p'}(\mathcal{N}) = \mathcal{M}(\lambda')$.

Proof. Let μ be either of the mappings λ or λ' . Then $\mu(U)$ is a subgroup of $N_{\mathcal{N}}(U)$ which is normal in $N_{\mathcal{L}}(O_p(\mathcal{L})U)$. The conditions (1) and (2) of 5.1 are immediate from I.2.3(b), so 5.1 yields $\mathcal{M}(\mu) \leq \mathcal{L}$.

Let \mathcal{N}_{Γ} and \mathcal{L}_0 be defined as in the first two paragraphs of the proof of 5.1, and set $\mathcal{M}_0 = \mathcal{M}(\mu) \cap \mathcal{N}_{\Gamma}$. Then $\mathcal{M}_0 \leq \mathcal{L}_0$, and \mathcal{M}_0 is the partial subgroup of \mathcal{L}_0 generated (in \mathcal{L}_0) by $\{\mu(U) \mid U \in \mathcal{E}^c\}$. As \mathcal{N}_{Γ} is a locality, we have the quotient locality $\overline{\mathcal{N}_{\Gamma}}/\mathcal{M}_0$, and the canonical projection $\rho : \mathcal{N}_{\Gamma} \rightarrow \overline{\mathcal{N}_G}$. As \mathcal{N}_{Γ} is a proper locality on \mathcal{E} , each element $g \in \mathcal{N}_{\Gamma}$ has an $\mathbf{A}_T(\mathcal{E})$ -decomposition $w = (g_1, \dots, g_n)$ (cf. 3.4). By definition, either $g_i \in O^p(N_{\mathcal{N}}(R_i)) \cap O^{p'}(N_{\mathcal{N}}(R_i))$ for some $R_i \in \mathcal{E}^{cr}$ (in which case $g_i \in \text{Ker}(\rho)$), or $g_i \in N_{\mathcal{N}}(T)$. It follows that $\overline{\mathcal{N}_G}$ is a p -group if $\mu = \lambda$, and a p' -group if $\mu = \lambda'$.

We provide the remaining details for the case $\mu = \lambda$. As $T\rho$ is a maximal p -subgroup of $\overline{\mathcal{N}_G}$ we obtain $\mathcal{N}_{\Gamma} = \mathcal{M}_0 T$. This shows that $O_{\mathcal{L}_0}^p(\mathcal{N}_{\Gamma}) \leq \mathcal{M}_0$. On the other hand, the image of \mathcal{N}_{Γ} in $\mathcal{L}_0/O_{\mathcal{L}_0}^p(\mathcal{N}_{\Gamma})$ is a p -group, so $O_{\mathcal{L}_0}^p(\mathcal{N}_{\Gamma})$ contains each of the groups $O^p(N_{\mathcal{N}}(U))$ for $U \in \Gamma$. Thus $O_{\mathcal{L}_0}^p(\mathcal{N}_{\Gamma}) = \mathcal{M}_0$. We now employ Theorem A in order to view \mathcal{L} as an expansion $(\mathcal{L}_0)^+$ of \mathcal{L}_0 , and to provide a correspondence between the partial normal subgroups of \mathcal{L}_0 and the partial normal subgroups of \mathcal{L} . Then lemma II.7.3 yields

$$O_{\mathcal{L}}^p(\mathcal{N}) = O_{\mathcal{L}_0}^p(\mathcal{N}_{\Gamma})^+ = \mathcal{M}_0(\lambda)^+.$$

Here $\mathcal{M}_0(\lambda)^+ = \mathcal{M}(\lambda)$ since $\mathcal{M}_0(\lambda) = \mathcal{M}(\lambda) \cap \mathcal{L}_0$, and this completes the proof in the case that $\mu = \lambda$. The proof for $\mu = \lambda'$ is essentially the same. \square

Notation 5.3. For elements $x, y \in \mathcal{L}$ write $[x, y] = \mathbf{1}$ if $S_{(x,y)} \cap S_{(y,x)} \in \Delta$ and $xy = yx$. For non-empty subsets X and Y of \mathcal{L} , write $[X, Y] = \mathbf{1}$ if $[x, y] = \mathbf{1}$ for all $x \in X$ and all $y \in Y$.

Lemma 5.4. Assume $\Sigma_T(\mathcal{F}) \subseteq \Delta$. Then the following hold.

- (a) $O_p(\mathcal{L}_T)C_{\mathcal{F}}(T)^{cr} \subseteq \Delta$. In particular, the hypothesis of 5.2 is fulfilled, with \mathcal{L}_T in the role of \mathcal{L} and with \mathcal{C}_T in the role of \mathcal{N} .
- (b) $O_{\mathcal{L}_T}^p(\mathcal{C}_T) = \langle O^p(N_{\mathcal{C}_T}(V)) \mid V \in C_{\mathcal{F}}(T)^{cr} \rangle$.
- (c) Let $X \in \Sigma_T(\mathcal{F})$ and let $V \in C_{\mathcal{F}}(T)^{cr}$. Then $[N_{\mathcal{N}}(X), O^p(N_{\mathcal{C}_T}(V))] = \mathbf{1}$.
- (d) $O_{\mathcal{L}_T}^p(\mathcal{C}_T)C_S(\mathcal{N}) \leq \mathcal{L}_T$.

Proof. Point (a) follows from the observation that $O_p(\mathcal{L})TC_{\mathcal{F}}(T)^{cr} \subseteq \Sigma_T(\mathcal{F})$ and that $T \leq O_p(\mathcal{L}_T)$. As $C_{\mathcal{F}}(T) = \mathcal{F}_{C_S(T)}(\mathcal{C}_T)$ by 1.5, we may apply 5.2 with \mathcal{L}_T and \mathcal{C}_T in the roles of \mathcal{L} and \mathcal{N} , and thereby obtain (b). Point (c) is given by 3.10. As $C_S(\mathcal{N}) \leq \mathcal{L}_T$ by 4.8, and $O_{\mathcal{L}_T}^p(\mathcal{C}_T) \leq \mathcal{L}_T$ by definition II.7.1, we have also point (d). \square

Definition 5.5. For any proper locality (\mathcal{L}, Δ, S) on \mathcal{F} , fix an expansion $(\mathcal{L}^s, \mathcal{F}^s, S)$ of \mathcal{L} to a proper locality on \mathcal{F} whose set of objects is the set \mathcal{F}^s of \mathcal{F} -subcentric subgroups of S . We shall refer to such a locality \mathcal{L}^s as a *subcentric closure* of \mathcal{L} . For any $\mathcal{N} \trianglelefteq \mathcal{L}$, let \mathcal{N}^s be the partial normal subgroup of \mathcal{L}^s whose intersection with \mathcal{L} is \mathcal{N} . Set

$$\mathcal{N}^\perp = O_{\mathcal{L}_T}^p(\mathcal{C}_T)C_S(\mathcal{N}^s),$$

where $T = S \cap \mathcal{N}$, $\mathcal{L}_T = N_{\mathcal{L}}(T)$, and $\mathcal{C}_T = C_{\mathcal{L}}(T)$. When no confusion is likely we shall write

$$T^\perp := S \cap \mathcal{N}^\perp,$$

even though T^\perp depends on \mathcal{N} rather than on T . The uniqueness of \mathcal{L}^s (up to a unique isomorphism which restricts to the identity map on \mathcal{L}), as given by Theorem II.A1, shows that $C_S(\mathcal{N}^s)$ does not depend on the choice of subcentric closure of \mathcal{L} , and thus \mathcal{N}^\perp and T^\perp are well-defined.

Lemma 5.6. Δ may be chosen so that $\Sigma_T(\mathcal{F}) \subseteq \Delta$, and so that:

- (*) For each $X \in \Sigma_T(\mathcal{F})$ such that $C_S(T) \leq X$, each $f \in N_{\mathcal{N}}(X)$, and each $w \in \mathbf{W}(\mathcal{C}_T) \cap \mathbf{D}$, we have $S_{(f) \circ w} \cap C_S(T)T \in \Delta$.

Proof. Assume to begin with that $\Delta = \mathcal{F}^s$. Set $\mathcal{M} = \langle \mathcal{C}_T \mathcal{N} \rangle$, $\tilde{T} = C_S(T)T$, $\mathcal{D} = \mathcal{F}_{\tilde{T}}(\mathcal{M})$, and $\Gamma = \mathcal{D}^c$. Then $\mathcal{M} \trianglelefteq \mathcal{L}$ and $S \cap \mathcal{M} = \tilde{T}$ by I.5.5. As $C_S(\mathcal{M}) \leq C_S(T) \leq \mathcal{M}$ we may appeal to 4.5 with \mathcal{M} in the role of \mathcal{N} , and conclude that $\Gamma \subseteq \Delta$, (\mathcal{M}, Γ) is a localizable pair, $(\mathcal{M}_\Gamma, \Gamma, \tilde{T})$ is a proper locality on \mathcal{D} , and that there is a natural conjugation action of H on \mathcal{D} . Then 4.1 applies to \mathcal{M}_Γ in the role of \mathcal{L} , and yields $\Sigma_T(\mathcal{F}) \subseteq \Gamma$. Let Δ_0 be the overgroup closure of Γ in S . As $\mathcal{L} = \mathcal{M}H$ by the Frattini Lemma, the action of H on \mathcal{D} implies that Δ_0 is \mathcal{F} -closed.

Note that, since \mathcal{M}_Γ is a proper locality on \mathcal{D} , \mathcal{D} is Γ -generated. Then by 2.8(a), with \mathcal{M} in the role of \mathcal{N} , we have $\mathcal{F}^{cr} \subseteq \Delta_0$, and the restriction \mathcal{L}_0 of \mathcal{L} to \mathcal{L}_0 is a proper locality on \mathcal{F} , with $\mathcal{M}_\Gamma = \mathcal{M} \cap \mathcal{L}_0$. Set $\mathcal{N}_0 = \mathcal{N} \cap \mathcal{L}_0$. Thus $\mathcal{N}_0 \trianglelefteq \mathcal{M}_\Gamma$, and it now suffices to show that (*) holds with \mathcal{L}_0 , \mathcal{N}_0 and $C_{\mathcal{L}_0}(T)$ in place of \mathcal{L} , \mathcal{N} and \mathcal{C}_T .

Let $X \in \Sigma_T(\mathcal{F})$ with $C_S(T) \leq X$, let each $f \in N_{\mathcal{N}_0}(X)$, and let $w \in \mathbf{W}(\mathcal{C}_{\mathcal{L}_0}(T)) \cap \mathbf{D}(\mathcal{L}_0)$. Set $U = X \cap T$ and $V = S_w \cap C_S(T)$. We now apply 4.1 to the locality \mathcal{M}_Γ in the role of \mathcal{L} (and with \tilde{T} in the role of S). Since $S_w \cap \tilde{T} = TV \in \mathcal{D}^c$, and $Z(T) \leq V$, it follows from 4.1(b) that $V \in C_{\mathcal{F}}(T)^c$. Since $U \in \mathcal{E}^c$ by 2.6(b), and since $\mathcal{E} = \mathcal{F}_T(\mathcal{N}_0)$ by 3.14, it follows from 4.1(a) that $UV \in \mathcal{D}^c$. Thus $UV \in \Delta_0$. Since $f^{-1} \in N_{\mathcal{N}}(X)$, and since $UV \leq UC_S(T) \leq X$, we have $(UV)^{f^{-1}} \leq S$. One then observes $(f) \circ w \in \mathbf{D}(\mathcal{L}_0)$ via $(UV)^{f^{-1}}$. Thus (*) holds, and the proof is complete. \square

Theorem 5.7. Let (\mathcal{L}, Δ, S) be a proper locality and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup.

- (a) Both $O_{\mathcal{L}_T}^p(\mathcal{C}_T)$ and \mathcal{N}^\perp are partial normal subgroups of \mathcal{L} .
(b) $S \cap \mathcal{N}^\perp = T^\perp \leq C_S(\mathcal{N})$, and if $\Sigma_T(\mathcal{F}) \subseteq \Delta$ then $T^\perp = C_S(\mathcal{N})$.

- (c) Let $f \in \mathcal{N}$ and $g \in \mathcal{N}^\perp$, and suppose that either $(f, g) \in \mathbf{D}$ or $(g, f) \in \mathbf{D}$. Then $[f, g] = \mathbf{1}$.
- (d) Let $(\mathcal{L}^+, \Delta^+, S)$ be an expansion of \mathcal{L} , and for any $\mathcal{K} \trianglelefteq \mathcal{L}$ let \mathcal{K}^+ be the unique partial normal subgroup of \mathcal{L}^+ (given by Theorem II.A2) whose intersection with \mathcal{L} is equal to \mathcal{K} . Then $(\mathcal{N}^+)^{\perp} = (\mathcal{N}^\perp)^+$.

Proof. By 5.6 there is a choice of Δ such that:

- (*) $\Sigma_T(\mathcal{F}) \subseteq \Delta$ and,
 (**) For each $X \in \Sigma_T(\mathcal{F})$ such that $C_S(T) \leq X$, each $f \in N_{\mathcal{N}}(X)$, and each $w \in \mathbf{W}(\mathcal{C}_T) \cap \mathbf{D}$, we have $S_{(f) \circ w} \in \Delta$.

Set $\mathcal{K} = O_{\mathcal{L}_T}^p(\mathcal{C}_T)$, and assume (*). Then, by 5.4(b), \mathcal{K} is generated (as a partial subgroup of \mathcal{L}) by the union Y_0 of the groups $O^p(N_{\mathcal{C}_T}(P))$ taken over all $P \in C_{\mathcal{F}}(T)^{cr}$. Recursively define Y_n for $n > 0$ to be the set of all $\Pi(w)$ with $w \in \mathbf{W}(Y_{n-1}) \cap \mathbf{D}$. Then \mathcal{K} is the union of the sets Y_n , by I.1.9. Now let $X \in \Sigma_T(\mathcal{F})$ with $C_S(T) \leq X$. Then $[N_{\mathcal{N}}(X), Y_0] = \mathbf{1}$ by 5.4(c). Let n be any index such that $[N_{\mathcal{N}}(X), Y_n] = \mathbf{1}$, let $f \in N_{\mathcal{N}}(X)$, and let $w = (g_1, \dots, g_k) \in \mathbf{W}(Y_n) \cap \mathbf{D}$. Assume now that (**) holds, so that $(f) \circ w \in \mathbf{D}$ via some $Q \leq C_S(T)T$. Set

$$w' = (f, g_1, f^{-1}, f, g_2, \dots, f^{-1}, f, g_k, f^{-1}, f).$$

Then $w' \in \mathbf{D}$ via Q . Since $g_i^{f^{-1}} = g_i$ for all i we obtain

$$fg = \Pi(f \circ w) = \Pi(w') = \Pi(w \circ f) = gf,$$

by \mathbf{D} -associativity (I.1.4(b)). Thus $[N_{\mathcal{N}}(X), Y_{n+1}] = \mathbf{1}$, and induction yields the following result.

- (1) If (*) and (**) hold then $[N_{\mathcal{N}}(X), \mathcal{K}] = \mathbf{1}$ for all $X \in \Sigma_T(\mathcal{F})$ such that $C_S(T) \leq X$.

Continue to assume (*) and (**), let f be an arbitrary element of \mathcal{N} , and let $u = (a, f_1, \dots, f_m)$ be an \mathcal{N} -decomposition of f . Then $a \in N_{\mathcal{N}}(\tilde{T})$ by 3.12, and so $a = tf_0$ where $t \in T$ and where $f_0 \in O^p(N_{\mathcal{N}}(\tilde{T}))$. Set $u' = (t, f_0, \dots, f_m)$, let $s \in S \cap \mathcal{K}$, and set

$$u'' = (s, t, s^{-1}, s, f_0, s^{-1}, \dots, s, f_m, s^{-1}).$$

Then $u'' \circ (s) \in \mathbf{D}$ via $(S_f)^{s^{-1}}$. Since $s \in \mathcal{K} \leq \mathcal{C}_T$, and since $(f_i)^s = f_i$ for all i with $0 \leq i \leq m$ by (1), we obtain

$$fs = \Pi(u \circ (s)) = \Pi(u'' \circ (s)) = \Pi((s) \circ u) = sf.$$

Thus $S \cap \mathcal{K} \leq C_S(\mathcal{N})$. By definition 5.5 we then have $S \cap \mathcal{N}^\perp = T^\perp = (S \cap \mathcal{K})C_S(\mathcal{N}^s)$. Here $C_S(\mathcal{N}) = C_S(\mathcal{N}^s)$ by (*) and 4.4, so we have shown:

- (2) If (*) and (**) hold then $T^\perp = C_S(\mathcal{N})$.

Let $(\mathcal{L}^+, \Delta^+, S)$ be an expansion of \mathcal{L} and let \mathcal{N}^+ be the unique partial normal subgroup of \mathcal{L}^+ whose intersection with \mathcal{L} is equal to \mathcal{N} . Then $C_S(\mathcal{N}) = C_S(\mathcal{N}^+)$ by 4.4. Thus $T^\perp = C_S(\mathcal{N}^+)$. Set $\mathcal{L}_T^+ = N_{\mathcal{L}^+}(T)$. Then $(\mathcal{L}_T^+, \Delta^+, S)$ is a proper locality on $N_{\mathcal{F}}(T)$ by 1.5(a), so $(\mathcal{L}_T^+, \Delta^+, S)$ is an expansion of the locality $(\mathcal{L}_T, \Delta, S)$, and $(\mathcal{N}^\perp)^+ \leq \mathcal{L}_T^+$. Observe that

$$(\mathcal{N}^\perp)^+ \mathcal{L}_T = (\mathcal{N}^\perp)^+ \mathcal{L} = \mathcal{N}^\perp.$$

Thus $(\mathcal{N}^\perp)^+$ is the partial normal subgroup of the locality \mathcal{L}_T^+ which corresponds to \mathcal{N}^\perp via Theorem II.A2. Recall that we have $\mathcal{N}^\perp = \mathcal{K}T^\perp$, where $\mathcal{K} = O_{\mathcal{L}_T}^p(\mathcal{C}_T)T^\perp$. Let \mathcal{K}^+ be the partial normal subgroup of \mathcal{L}_T corresponding to \mathcal{K} . Then $\mathcal{K}^+ = O_{\mathcal{L}_T^+}^p(C_{\mathcal{L}^+}(T))$ by II.7.3, and then $\mathcal{K}^+T^\perp \leq \mathcal{L}_T^+$ by 5.4(d). Then, since $(\mathcal{K}^+T^\perp) \cap \mathcal{L} = \mathcal{N}^\perp$, we conclude that $(\mathcal{N}^\perp)^+ = \mathcal{K}^+T^\perp$. Then $(\mathcal{N}^\perp)^+ = (\mathcal{N}^+)^+^\perp$ by definition 5.5.

Now let $(\mathcal{L}', \Delta', S)$ be the restriction of \mathcal{L}^+ to a locality (necessarily proper, by II.2.11) on an \mathcal{F} -closed subset Δ' of \mathcal{F}^s containing \mathcal{F}^{cr} . For any partial subgroup \mathcal{H} of \mathcal{L}^+ set $\mathcal{H}' = \mathcal{H} \cap \mathcal{L}'$. By a straightforward exercise with definition II.7.1 one has $\mathcal{K}' = O_{(\mathcal{L}_T^+)' }^p(C_{\mathcal{L}^+}(T)')$, and hence $(\mathcal{N}^\perp)' = (\mathcal{N}')^\perp$. Since any proper locality on \mathcal{F} can be obtained (up to isomorphism) from \mathcal{L} by a procedure of first expanding and then restricting, by II.A1, we thereby obtain (d). By 4.4 and (2), we obtain also point (b). In particular (and without recourse to (*) or (**)):

$$(3) \quad T^\perp \leq C_S(\mathcal{N}).$$

Assume that (*) holds. Then $\Sigma_{C_S(T)}(\mathcal{C}_T) \subseteq \Delta$ by 3.9(c), and so every element of \mathcal{L}_T has a \mathcal{C}_T -decomposition by 3.12. Let $g \in \mathcal{N}^\perp$, and let $v = (b, g_1, \dots, g_n)$ be a \mathcal{C}_T -decomposition of g . Write $b = sg_0$ where $s \in C_S(T)$ and $g_0 \in O^p(N_{\mathcal{K}}(T^\perp))$. Set $v' = (s, g_0, \dots, g_n)$. Then $S_v = S_{v'}$ and $g = \Pi(v')$. As $g \in \mathcal{N}^\perp$ and $g_0 \cdots g_n \in \mathcal{K} \leq \mathcal{N}^\perp$, it follows that $s \in S \cap \mathcal{N}^\perp$. Thus $s \in C_S(\mathcal{N})$ by (3). Let $f \in \mathcal{N}$, and let $u = (f_1, \dots, f_m)$ by an \mathcal{N} -decomposition of f . Suppose that $(f, g) \in \mathbf{D}$. Then $u \circ v' \in \mathbf{D}$, and $fg = \Pi(u \circ v')$. Since s commutes with each f_i we have

$$fg = s\Pi(f_1, \dots, f_m, g_0, \dots, g_n).$$

Since each $[f_i, g_j] = \mathbf{1}$ for all i and j , by 5.4(c), we have $S_{(f_i, g_j)} = S_{(g_j, f_i)}$ and $P^{f_i g_j} = P^{g_j f_i}$ for all $P \in \Delta$ with $P \leq S_{(f_i, g_j)}$. From this fact, a straightforward argument by induction on $m + n$ yields:

$$fg = s\Pi(g_1, \dots, g_n, f_0, \dots, f_m),$$

and then $fg = gf$. Evidently the assumption $(f, g) \in \mathbf{D}$ can be replaced by the symmetric assumption $(g, f) \in \mathbf{D}$, so:

$$(4) \quad \text{If } \Sigma_T(\mathcal{F}) \subseteq \Delta \text{ then (c) holds.}$$

Since $\mathcal{L} = \mathcal{N}\mathcal{L}_T$, and since $\mathcal{N}^\perp \leq \mathcal{L}_T$, one may employ (4) and the splitting lemma (I.3.12) to obtain $\mathcal{N}^\perp \leq \mathcal{L}$ in the case that $\Sigma_T(\mathcal{F}) \subseteq \Delta$. But then $\mathcal{N}^\perp \leq \mathcal{L}$ in general, by (d). Similarly, since $\mathcal{K} \leq \mathcal{N}^\perp$ and $\mathcal{K} \leq \mathcal{L}_T$, we obtain $\mathcal{K} \leq \mathcal{L}$. Thus (a) holds, and the proof is complete. \square

Section 6: $\partial(\mathcal{F})$, $\delta(\mathcal{F})$, and $F^*(\mathcal{L})$

As always, (\mathcal{L}, Δ, S) is a proper locality on \mathcal{F} . We are now in position to consider all of the partial normal subgroups \mathcal{N} of \mathcal{L} simultaneously. And as always, there is the difficulty in the back-ground, that the fusion system \mathcal{E} of \mathcal{N} need not be stable with respect to the processes of expansion (given by Theorem II.A) and restriction (given by II.2.11). Since $S \cap \mathcal{N}$ is stable with respect to these two processes, one step towards addressing the difficulty is with a simple notational device, as follows.

For $\mathcal{N} \trianglelefteq \mathcal{L}$, and $T = S \cap \mathcal{N}$, define $\mathcal{E}^s(\mathcal{N})$ to be the fusion system on T of the form $\mathcal{F}_T(\mathcal{N}^s)$; where \mathcal{N}^s is the partial normal subgroup of an expansion $(\mathcal{L}^s, \mathcal{F}^s, S)$ of \mathcal{L} (as in Theorem II.A1) whose intersection with \mathcal{L} is \mathcal{N} (as in Theorem II.A2). By 3.14, $\mathcal{E}^s(\mathcal{N})$ is stable with respect to restriction from \mathcal{F}^s to \mathcal{F} -closed sets Δ containing $\Sigma_T(\mathcal{F})$.

Definition 6.1. A partial normal subgroup $\mathcal{M} \trianglelefteq \mathcal{L}$ will be said to be *large* in \mathcal{L} if $S \cap \mathcal{M}^\perp \leq \mathcal{M}$. Let $\mathbb{M} := \mathbb{M}(\mathcal{L})$ be the set of all large partial normal subgroups of \mathcal{L} . Define $\partial(\mathcal{F})$ to be the overgroup closure in S of $\bigcup \{O_p(\mathcal{L})\mathcal{E}^s(\mathcal{M})^{cr} \mid \mathcal{M} \in \mathbb{M}\}$.

Lemma 6.2.

- (a) $\partial(\mathcal{F})$ depends only on \mathcal{F} (and not on the choice of a proper locality on \mathcal{F}).
- (b) $\partial(\mathcal{F})$ is \mathcal{F} -closed, and $\mathcal{F}^{cr} \subseteq \partial(\mathcal{F}) \subseteq \mathcal{F}^s$.
- (c) Let $\mathcal{M} \in \mathbb{M}$ and set $\mathcal{D} = \mathcal{E}^s(\mathcal{M})$. Then \mathcal{D} is (cr) -generated, and $\mathcal{D}^{cr} \subseteq \mathcal{F}^q$.

Proof. By Theorem II.A1, any proper locality \mathcal{L}' on \mathcal{F} is isomorphic to a “version” of \mathcal{L} obtained by first expanding Δ to \mathcal{F}^s , and then restricting to an \mathcal{F} -closed subset Δ' of \mathcal{F}^s containing \mathcal{F}^{cr} . By Theorem II.A2 there is a canonical isomorphism $\mathcal{N} \mapsto \mathcal{N}'$ from the poset of partial normal subgroups of \mathcal{L} to that of \mathcal{L}' , and 5.5(c) implies that $(\mathcal{N}^\perp)' = (\mathcal{N}')^\perp$. Then $S \cap \mathcal{N}^\perp = S \cap (\mathcal{N}')^\perp$ (by II.A2). As $O_p(\mathcal{L}) = O_p(\mathcal{L}')$ by II.2.3, we obtain (a).

Let $\mathcal{M} \in \mathbb{M}$, set $R = S \cap \mathcal{M}$, and set $\mathcal{D} = \mathcal{E}^s(\mathcal{M})$. Let \mathcal{L}^+ be the expansion of \mathcal{L} to $\Delta^+ := \mathcal{F}^s$, and let \mathcal{M}^+ be the partial normal subgroup of \mathcal{L}^+ whose intersection with \mathcal{L} is \mathcal{M} . Then $\mathcal{D} = \mathcal{F}_R(\mathcal{M}^+)$ by definition. We have

$$C_S(\mathcal{M}^+) = S \cap (\mathcal{M}^+)^\perp = S \cap (\mathcal{M}^\perp)_+ = S \cap \mathcal{M}$$

by 5.5, so $C_S(\mathcal{M}^+) \leq \mathcal{M}^+$.

Let Γ be the overgroup closure of \mathcal{D}^{cr} in R , and let Δ_0 be the overgroup closure of $O_p(\mathcal{L})\Gamma$ in S . Then $\Gamma \subseteq \mathcal{F}^q$ by 4.3(c), so $\Delta_0 \subseteq \Delta^+$, and thus $\partial(\mathcal{F}) \subseteq \mathcal{F}^s$. We have $\mathcal{F}^{cr} \subseteq \partial(\mathcal{F})$ as a consequence of 2.6(b). As $R \in \Gamma \subseteq \mathcal{F}^q$, \mathcal{D} is (cr) -generated by 4.3(d), and thus (c) holds. Then 2.8 applies with \mathcal{M} in the role of \mathcal{N} , and 2.8(c) shows that Γ is invariant under the action of $N_{\mathcal{L}}(R)$. By the splitting lemma (I.3.12) Γ is then \mathcal{F} -invariant, so Δ_0 is \mathcal{F} -invariant. As $\partial(\mathcal{F})$ is the union of the various sets Δ_0 taken over all $\mathcal{M} \in \mathbb{M}$, the proof of (b) is complete. \square

Lemma 6.3. Let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} , set $T = S \cap \mathcal{N}$, and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$. Further, set $T^\perp = S \cap \mathcal{N}^\perp$ and $\mathcal{E}^\perp = \mathcal{F}_{T^\perp}(\mathcal{N}^\perp)$. Then $\Sigma_T(\mathcal{F}) \subseteq \partial(\mathcal{F})$, and if $\partial(\mathcal{F}) \subseteq \Delta$ then:

- (a) $O_p(\mathcal{L})\mathcal{E}^{cr}(\mathcal{E}^\perp)^{cr} \subseteq \partial(\mathcal{F})$.

(b) \mathcal{E} is (cr) -generated.

Proof. We begin by proving (a) and (b) under the assumption that $\Sigma_{S \cap \mathcal{K}}(\mathcal{F}) \subseteq \Delta$ for all $\mathcal{K} \trianglelefteq \mathcal{L}$. Thus, set $\mathcal{M} = \mathcal{N}\mathcal{N}^\perp$ and $R = TT^\perp$. As $\Sigma_T(\mathcal{F}) \subseteq \Delta$ by assumption, 5.7(b) yields $T^\perp = C_S(\mathcal{N})$, and then I.5.1 yields $\mathcal{M} \trianglelefteq \mathcal{L}$ and $R = S \cap \mathcal{M}$. Set $\mathcal{D} = \mathcal{F}_R(\mathcal{M})$. As $\Sigma_R(\mathcal{M}) \subseteq \Delta$ we have $\mathcal{D} = \mathcal{E}^s(\mathcal{M})$ by 3.14. Note that

$$S \cap \mathcal{M}^\perp = C_S(\mathcal{M}) \leq C_S(\mathcal{N}) = T^\perp \leq \mathcal{M},$$

so that $\mathcal{M} \in \mathbb{M}$. Then \mathcal{D} is (cr) -generated and $\mathcal{D}^{cr} \subseteq \Delta$ by 6.2(c). We may now appeal to 2.10, with $\mathcal{N}_1 = \mathcal{N}$ and $\mathcal{N}_2 = \mathcal{N}^\perp$, to obtain

$$\mathcal{D}^{cr} = \mathcal{E}^{cr}(\mathcal{E}^\perp)^{cr}$$

via 2.10(d). This yields (a). By 2.10(c) \mathcal{E} is the fusion system of a proper locality, and then (b) follows from II.2.10.

Assume now that $\Delta = \mathcal{F}^s$. Then $\Sigma_{S \cap \mathcal{K}}(\mathcal{F}) \subseteq \Delta$ for all $\mathcal{K} \trianglelefteq \mathcal{L}$, and we may make further use of 2.10. Namely, 2.10(c) yields:

(*) Both \mathcal{E}^{cr} and $(\mathcal{E}^\perp)^{cr}$ are $N_{\mathcal{L}}(R)$ -invariant.

Let $V \in C_{\mathcal{F}}(T)^{cr}$. By II.1.15 there exists an $N_{\mathcal{F}}(T)$ -conjugate V' of V such that $X' := V' \cap T^\perp$ is fully normalized in $N_{\mathcal{F}}(T)$. Then X' is fully normalized in $C_{\mathcal{F}}(T)$ by II.1.17. Set $\mathcal{L}_T = N_{\mathcal{L}}(T)$ and $\mathcal{C}_T = C_{\mathcal{L}}(T)$.

Note that $C_{\mathcal{F}}(T)^{cr}$ is $N_{\mathcal{L}}(R)$ -invariant, by an application of 2.10(c) to the partial normal subgroup TC_T of the locality \mathcal{L}_T . Thus $V' \in C_{\mathcal{F}}(T)^{cr}$. As $C_{\mathcal{F}}(T) = \mathcal{F}_{C_S(T)}(C_{\mathcal{L}}(T))$ by 1.4, and since $C_{\mathcal{F}}(T)$ is inductive by II.6.1, the hypothesis of 2.6(c) is fulfilled with $(\mathcal{L}_T, \mathcal{C}_T, \mathcal{N}^\perp)$ in the place of $(\mathcal{L}, \mathcal{M}, \mathcal{N})$. We therefore conclude that $X' \in (\mathcal{E}^\perp)^{cr}$. Set $X = V \cap T^\perp$. Then X is an $N_{\mathcal{F}}(T)$ -conjugate of X' . As $N_{\mathcal{F}}(T) = \mathcal{F}_S(\mathcal{L}_T)$, and $\mathcal{L}_T = \mathcal{E}^\perp N_{\mathcal{L}_T}(R)$ by the Frattini Lemma, it follows from (*) that $X \in (\mathcal{E}^\perp)^{cr}$. Thus:

(**) $V \cap T^\perp \in (\mathcal{E}^\perp)^{cr}$ for all $V \in C_{\mathcal{F}}(T)^{cr}$.

Now let $Y \in \Sigma_T(\mathcal{F})$, and write $Y = O_p(\mathcal{L})UV$ as in 3.7. Notice that since $T^\perp = S \cap \mathcal{E}^\perp$ is weakly closed in \mathcal{F} we have $N_{\mathcal{L}}(C_S(T)T) \leq N_{\mathcal{L}}(R)$. As \mathcal{E}^{cr} is $N_{\mathcal{L}}(R)$ -invariant by (*), we obtain $U \in \mathcal{E}^{cr}$ from 2.6(b). Then $U(V \cap T^\perp) \in \mathcal{D}^{cr}$ by 2.10(d), and thus $Y \in \partial(\mathcal{F})$. Having thus shown that $\Sigma_T(\mathcal{F}) \subseteq \partial(\mathcal{F})$, the proof of the lemma is complete. \square

Our next aim is to show that the intersection of the set of large partial normal subgroups of \mathcal{L} containing $O_p(\mathcal{L})$ is itself large. Lemmas 6.4 through 6.7 will achieve that result.

Recall that for any partial subgroup $\mathcal{H} \leq \mathcal{L}$, $Z(\mathcal{H})$ is defined to be the set of all $z \in \mathcal{H}$ such that, for all $h \in \mathcal{H}$, h^z is defined and is equal to h .

Lemma 6.4. *Assume $\partial(\mathcal{F}) \subseteq \Delta$, let \mathcal{N} be a partial normal subgroup of \mathcal{L} , and set $T = S \cap \mathcal{N}$. Then $T^\perp = C_S(\mathcal{N})$, and $Z(\mathcal{N}) = C_T(\mathcal{N})$. Moreover, the following are equivalent.*

- (1) $T^\perp = Z(\mathcal{N})$.
- (2) $T^\perp \leq \mathcal{N}$.
- (3) $\mathcal{N}^\perp = Z(\mathcal{N})$.

Proof. We have $\Sigma_T(\mathcal{F}) \subseteq \Delta$ by 6.3. Then $T^\perp = C_S(\mathcal{N})$ by definition, and $Z(\mathcal{N}) = C_T(\mathcal{N})$ by 4.8. The implication (3) \implies (1) is then immediate. Since the implication (1) \implies (2) is trivial, it remains only to show that (2) \implies (3).

Assume (2). Then $T^\perp = Z(\mathcal{N})$, and so \mathcal{E}^\perp is the trivial fusion system on an abelian p -group. As \mathcal{E}^\perp is the fusion system of \mathcal{N}^\perp , where $\mathcal{N}^\perp = O_{\mathcal{L}_T}^p(\mathcal{C}_T)T^\perp$, it follows that $\mathcal{N}^\perp = T^\perp$, and thus (3) holds. \square

Next, recall from II.7.2 that for any $\mathcal{N} \trianglelefteq \mathcal{L}$, $O_{\mathcal{L}}^p(\mathcal{N})$ is the smallest partial normal subgroup $\mathcal{K} \trianglelefteq \mathcal{L}$ such that $\mathcal{N} = (S \cap \mathcal{N})\mathcal{K}$.

Lemma 6.5. *Assume $\partial(\mathcal{F}) \subseteq \Delta$, and let \mathcal{M} and \mathcal{N} be partial normal subgroups of \mathcal{L} . Suppose that $\mathcal{M} \cap \mathcal{N} \leq Z(\mathcal{N})$. Then $O_{\mathcal{L}}^p(\mathcal{N}) \leq \mathcal{M}^\perp$.*

Proof. Set $R = S \cap \mathcal{M}$, $T = S \cap \mathcal{N}$, and $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$. Then $[R, T] \leq \mathcal{M} \cap \mathcal{N}$, so $[R, T] \leq Z(\mathcal{N})$ by hypothesis. Let $U \in \mathcal{E}^{cr}$. Then R normalizes U , and then R normalizes UT^\perp since $T^\perp := S \cap \mathcal{N}^\perp \trianglelefteq S$. Moreover, we have $UT^\perp \in \partial(\mathcal{F})$ by 6.3(a), so $UT^\perp \in \Delta$, and thus both R and $N_{\mathcal{N}}(U)$ are subgroups of the group $N_{\mathcal{L}}(UT^\perp)$. As $Z(\mathcal{N}) \leq O_p(\mathcal{N})$ by 2.7, we obtain

$$[R, N_{\mathcal{N}}(U)] \leq Z(\mathcal{N}) \leq U,$$

and hence $[R, O^p(N_{\mathcal{N}}(U))] = \mathbf{1}$ by II.2.7(c).

Set $\mathcal{L}_R = N_{\mathcal{L}}(R)$ and $\mathcal{C}_R = C_{\mathcal{L}}(R)$. Then $\mathcal{C}_R = O_{\mathcal{L}_R}^p(\mathcal{C}_R)C_S(R)$. The canonical projection $\mathcal{L}_R \rightarrow \mathcal{L}_R/O_{\mathcal{L}_R}^p(\mathcal{C}_R)$ maps $O^p(N_{\mathcal{N}}(U))$ to a p -group, and so $O^p(N_{\mathcal{N}}(U)) \leq O_{\mathcal{L}_R}^p(\mathcal{C}_R)$. Thus $O^p(N_{\mathcal{N}}(U)) \leq \mathcal{M}^\perp$ for all $U \in \mathcal{F}^{cr}$, and the lemma now follows from 5.2. \square

Lemma 6.6. *Assume $\partial(\mathcal{F}) \subseteq \Delta$, and let \mathcal{M} and \mathcal{N} be partial normal subgroups of \mathcal{L} . Assume that $C_S(\mathcal{M}) \leq \mathcal{M}$ and that $\mathcal{M} \cap \mathcal{N}^\perp \leq Z(\mathcal{N})$. Then $\mathcal{N}^\perp = C_S(\mathcal{N})$.*

Proof. Since $Z(\mathcal{N}) \leq Z(\mathcal{N}^\perp)$, the hypothesis that $\mathcal{M} \cap \mathcal{N}^\perp$ be contained in $Z(\mathcal{N})$ enables an application of the preceding lemma with \mathcal{N}^\perp in the role of \mathcal{N} . Thus $O_{\mathcal{L}}^p(\mathcal{N}^\perp) \leq \mathcal{M}^\perp$. Since $C_S(\mathcal{M}) \leq \mathcal{M}$, 6.4 yields $\mathcal{M}^\perp = Z(\mathcal{M}) = C_S(\mathcal{M})$, and thus $O_{\mathcal{L}}^p(\mathcal{N}^\perp)$ is a p -group. Since $S \cap \mathcal{N}^\perp = C_S(\mathcal{N})$ by 6.6, the lemma follows. \square

Lemma 6.7. *Assume $\partial(\mathcal{F}) \subseteq \Delta$, and let \mathcal{M} and \mathcal{N} be partial normal subgroups of \mathcal{L} . Assume that $O_p(\mathcal{L})C_S(\mathcal{M}) \leq \mathcal{M}$ and that $O_p(\mathcal{L})C_S(\mathcal{N}) \leq \mathcal{N}$. Then $O_p(\mathcal{L})C_S(\mathcal{M} \cap \mathcal{N}) \leq \mathcal{M} \cap \mathcal{N}$.*

Proof. Set $\mathcal{K} = \mathcal{M} \cap \mathcal{N}$. Obviously $O_p(\mathcal{L}) \leq \mathcal{K}$, so the problem is to show that $C_S(\mathcal{K}) \leq \mathcal{K}$. Observe:

$$\mathcal{M} \cap (\mathcal{N} \cap \mathcal{K}^\perp) = \mathcal{K} \cap \mathcal{K}^\perp = Z(\mathcal{K}) \leq Z(\mathcal{K}^\perp).$$

Since $Z(\mathcal{K}) \leq \mathcal{N}$ we then have $Z(\mathcal{K}) \leq Z(\mathcal{N} \cap \mathcal{K}^\perp)$, and we may apply 6.5 with $\mathcal{N} \cap \mathcal{K}^\perp$ in the role of \mathcal{N} . Thus $O_{\mathcal{L}}^p(\mathcal{N} \cap \mathcal{K}^\perp) \leq \mathcal{M}^\perp$. As $C_S(\mathcal{M}) \leq \mathcal{M}$, $\mathcal{M}^\perp = C_S(\mathcal{M})$ by 6.4. Thus $\mathcal{N} \cap \mathcal{K}^\perp$ is a p -group, and indeed a normal p -subgroup of \mathcal{L} . As $O_p(\mathcal{L}) \leq \mathcal{K}$, we obtain $\mathcal{N} \cap \mathcal{K}^\perp \leq \mathcal{K}$, and so $\mathcal{N} \cap \mathcal{K}^\perp \leq Z(\mathcal{K})$. Now the hypothesis of 6.6 is satisfied, with \mathcal{N} in the role of \mathcal{M} , and with \mathcal{K} in the role of \mathcal{N} . We conclude that $\mathcal{K}^\perp = C_S(\mathcal{K})$. As $\mathcal{K}^\perp \trianglelefteq \mathcal{L}$, and $O_p(\mathcal{L}) \leq \mathcal{K}$, we obtain $C_S(\mathcal{K}) \leq \mathcal{K}$, as required. \square

Definition 6.8. If $\partial(\mathcal{F}) \subseteq \Delta$, define $F^*(\mathcal{L})$ to be the intersection of the large partial normal subgroups of \mathcal{L} containing $O_p(\mathcal{L})$:

$$\mathcal{F}^*(\mathcal{L}) = \bigcap \{ \mathcal{N} \mid O_p(\mathcal{L})C_S(\mathcal{N}) \leq \mathcal{N} \trianglelefteq \mathcal{L} \}.$$

More generally, define $F^*(\mathcal{L})$ to be $F^*(\mathcal{L}^+) \cap \mathcal{L}$, where $(\mathcal{L}^+, \Delta^+, S)$ is the expansion of \mathcal{L} to a proper locality on \mathcal{F} with $\Delta^+ = \Delta \cup \partial(\mathcal{F})$.

Corollary 6.9. *If $\partial(\mathcal{F}) \subseteq \Delta$ then $F^*(\mathcal{L})$ is a large partial normal subgroup of \mathcal{L} containing $O_p(\mathcal{L})$, and is the unique smallest such. Moreover, and in general (i.e. whether or not $\partial(\mathcal{F}) \subseteq \Delta$), we have $F^*(\mathcal{L}^+) = F^*(\mathcal{L})^+$ for any expansion $(\mathcal{L}^+, \Delta^+, S)$ of \mathcal{L} to a proper locality on \mathcal{F} . \square*

Proof. The first assertion is immediate from 6.7. Now drop the assumption that $\partial(\mathcal{F})$ is contained in Δ , and let $(\mathcal{L}^+, \Delta^+, S)$ be an expansion of \mathcal{L} to some Δ^+ with $\mathcal{D} \cup \partial(\mathcal{F}) \subseteq \Delta^+ \subseteq \mathcal{F}^s$. Let $(\mathcal{L}^s, \mathcal{F}^s, \Sigma)$ be the expansion of \mathcal{L} to \mathcal{F}^s . Let \mathbf{X}^+ be the set of large partial normal subgroups \mathcal{N}^+ of \mathcal{L}^+ containing $O_p(\mathcal{F})$, and similarly define \mathbf{X}^s relative to \mathcal{L}^s .

By 6.3 we have $\Sigma_T(\mathcal{F}) \subseteq \partial(\mathcal{F})$ for all $T = S \cap \mathcal{N}$ with $\mathcal{N} \trianglelefteq \mathcal{L}$. Then by 3.14, the correspondence (Theorem II.A2) between the set of partial normal subgroups of \mathcal{L}^+ and the set of partial normal subgroups of \mathcal{L}^s restricts to a bijection $\mathbf{X}^+ \rightarrow \mathbf{X}^s$. Then

$$F^*(\mathcal{L}^+) = \bigcap \mathbf{X}^+ = \mathcal{L}^+ \cap \left(\bigcap \mathbf{X}^s \right) = \mathcal{L}^+ \cap F^*(\mathcal{L}^s).$$

This shows that $F^*(\mathcal{L}^+)$ is determined by $F^*(\mathcal{L}^s)$. Then

$$F^*(\mathcal{L}) = \mathcal{L} \cap F^*(\mathcal{L}^s) = \mathcal{L} \cap F^*(\mathcal{L}^+).$$

The correspondence given by Theorem II.A2 then yields $F^*(\mathcal{L}^+) = F^*(\mathcal{L}^+)$. \square

Set $F^*(\mathcal{F}) = \mathcal{F}_{S \cap F^*(\mathcal{L})}(F^*(\mathcal{L}))$, and define $\delta(\mathcal{F})$ to be the overgroup-closure of $F^*(\mathcal{F})^s$ in S .

Lemma 6.10. *$\delta(\mathcal{F})$ depends only on \mathcal{F} (and not on the choice of proper locality on \mathcal{F}). Moreover, $\delta(\mathcal{F})$ is \mathcal{F} -closed, and*

$$\mathcal{F}^{cr} \subseteq \partial(\mathcal{F}) \subseteq \delta(\mathcal{F}) \subseteq \mathcal{F}^s.$$

Proof. As $\partial(\mathcal{F})$ depends only on \mathcal{F} by 6.2(a), and since \mathcal{L}^s depends only on \mathcal{F} by Theorem II.A1, it follows that $\delta(\mathcal{F})$ depends only on \mathcal{F} . Now let \mathcal{M} be an arbitrary large partial normal subgroup of \mathcal{L} containing $O_p(\mathcal{L})$, set $R = S \cap \mathcal{M}$, and set $\mathcal{D} = \mathcal{F}_R(\mathcal{M})$. As $\Sigma_R(\mathcal{F}) \subseteq \partial(\mathcal{F})$ by 6.3, it follows from 3.14 that \mathcal{D} is independent of the choice of Δ , provided only that $\partial(\mathcal{F}) \subseteq \Delta \subseteq \mathcal{F}^s$. Taking $\Delta = \mathcal{F}^s$, we obtain $\mathcal{D}^c \subseteq \mathcal{F}^q$ by 4.5(a), so $R \in \mathcal{F}^q$, and then $\mathcal{D}^s \subseteq \mathcal{F}^s$ by 4.3(c). In particular, by taking $\mathcal{M} = F^*(\mathcal{L})$, we obtain $F^*(\mathcal{F})^s \subseteq \mathcal{F}^s$.

As $\mathcal{D}^{cr} \subseteq \Delta$, it follows from 2.8 that \mathcal{D}^s is \mathcal{F} -invariant, and hence the overgroup-closure of \mathcal{D}^s in S is \mathcal{F} -closed. By specializing again to $\mathcal{M} = F^*(\mathcal{L})$, we conclude that $\delta(\mathcal{F})$ is \mathcal{F} -closed. As $\mathcal{F}^{cr} \subseteq \partial(\mathcal{F})$ by 6.2(b), it now only remains to show that $\partial(\mathcal{F}) \subseteq \delta(\mathcal{F})$.

Let $P \in \mathcal{D}^{cr}$. Then there exists $Q \in P^{\mathcal{F}}$ such that $P \cap F^*(\mathcal{L})$ is fully normalized in \mathcal{F} , by II.1.15. We note that \mathcal{D}^{cr} is \mathcal{F} -invariant (again by 2.8), so $Q \in \mathcal{D}^{cr}$, and then $Q \cap F^*(\mathcal{L}) \in F^*(\mathcal{F})^{cr}$ by 2.6(c). Thus $Q \in \delta(\mathcal{F})$, and then $P \in \delta(\mathcal{F})$ as $\delta(\mathcal{F})$ is \mathcal{F} -invariant. Thus $\partial(\mathcal{F}) \subseteq \delta(\mathcal{F})$, and the proof is complete. \square

Section 7: Regular localities

In the preceding section we produced a mapping δ which, to each saturated fusion system \mathcal{F} on a p -group S , assigns the overgroup closure in S of $\mathcal{F}^*(\mathcal{F})^s$. By 6.8(a) and Theorem II.A1 there is a unique (up to unique isomorphism) proper locality \mathcal{L} on \mathcal{F} whose set of objects is $\delta(\mathcal{F})$.

Definition 7.1. A locality (\mathcal{L}, Δ, S) on \mathcal{F} is *regular* provided that \mathcal{L} is proper, and $\Delta = \delta(\mathcal{F})$.

Proposition 7.2. Let (\mathcal{L}, Δ, S) be a regular locality, let $Z \leq Z(\mathcal{L})$ be a subgroup of $Z(\mathcal{L})$, and let $\rho : \mathcal{L} \rightarrow \mathcal{L}/Z$ be the canonical projection. Then \mathcal{L}/Z is regular, and $F^*(\mathcal{L})\rho = \mathcal{F}^*(\mathcal{L}/Z)$.

Proof. Set $\overline{\mathcal{L}} = \mathcal{L}/Z$, and write \overline{X} for the image under ρ of any subset of \mathcal{L} , or any set of subgroups of \mathcal{L} . Set $\mathcal{F} = \mathcal{F}_S(\mathcal{L})$ and set $\overline{\mathcal{F}} = \mathcal{F}_{\overline{S}}(\overline{\mathcal{L}})$. The restriction $\sigma : S \rightarrow \overline{S}$ of ρ to S is fusion-preserving, and we shall denote also by σ the associated homomorphism $\mathcal{F} \rightarrow \overline{\mathcal{F}}$ of fusion systems. Then σ maps the set $Hom(\mathcal{F})$ of \mathcal{F} -homomorphisms onto $Hom(\overline{\mathcal{F}})$, so II.1.19 applies and yields the following information concerning a subgroup \overline{X} of \overline{S} and its preimage X in S .

- (1) X is fully normalized in \mathcal{F} if and only if \overline{X} is fully normalized in $\overline{\mathcal{F}}$.
- (2) $\overline{O_p(N_{\mathcal{F}}(X))} \leq O_p(N_{\overline{\mathcal{F}}}(\overline{X}))$.
- (3) If $\overline{X} \in \overline{\mathcal{F}}^c$ then $X \in \mathcal{F}^c$, and if $\overline{X} \in \overline{\mathcal{F}}^{cr}$ then $X \in \mathcal{F}^{cr}$.

As a consequence of (3) we have $\overline{\mathcal{F}}^{cr} \leq \overline{\Delta}$. For any $P \in \Delta$ the group $N_{\mathcal{L}}(P)/Z$ is of characteristic p , by II.2.6(c), so $\overline{\mathcal{L}}$ is a proper locality. We next show:

- (4) $\overline{\mathcal{F}}^s = \overline{\mathcal{F}}^s$.

In one direction: let $Z \leq P \leq S$ with P fully normalized in \mathcal{F} and with $\overline{P} \in \overline{\mathcal{F}}^s$. Let Q be the pre-image in S of $O_p(N_{\overline{\mathcal{L}}}(\overline{P}))$. Then $Q = O_p(N_{\mathcal{L}}(P))$, and $Q \in \mathcal{F}^c$ by (3). Thus $\overline{\mathcal{F}}^s \supseteq \overline{\mathcal{F}}^s$. In the other direction: let $P \in \mathcal{F}^s$. Then $ZP \in \mathcal{F}^s$, as is any \mathcal{F} -conjugate of P . In order to show that $\overline{P} \in \overline{\mathcal{F}}^s$ we may therefore assume that $Z \leq P$, that P is fully normalized in \mathcal{F} , and (by II.1.18) that $Q := O_p(N_{\mathcal{F}}(P))$ is fully normalized in \mathcal{F} . Let D be the pre-image of $C_{\overline{S}}(\overline{Q})$ in S . Thus $[Q, D] \leq Z$. Let $(\mathcal{L}^s, \mathcal{F}^s, S)$ be the expansion of \mathcal{L} to a proper locality on \mathcal{F} whose set of objects is \mathcal{F}^s , and set $M = N_{\mathcal{L}^s}(P)$. Thus M is a subgroup of \mathcal{L}^s of characteristic p . Here \mathcal{L}^s is generated by \mathcal{L} as a partial group, by point (c) in Theorem II.A1, so $Z \leq Z(\mathcal{L}^s)$. Thus $Z \leq Z(M)$, and D centralizes the

chain $Q \geq Z \geq 1$ of normal subgroups of M . Now II.2.6(c) shows that $D = Q$, and thus \overline{Q} is centric in $\overline{\mathcal{F}}$. Thus (4) holds. We next show:

- (5) Let $\mathcal{M} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} containing $O_p(\mathcal{L})$. Then \mathcal{M} is large in \mathcal{L} if and only if \mathcal{M}/Z is large in \mathcal{L}/Z .

Evidently $C_S(\mathcal{M})/Z \leq C_{\overline{S}}(\overline{\mathcal{M}})$, so if \mathcal{M}/Z is large in \mathcal{L}/Z then \mathcal{M} is large in \mathcal{L} . Conversely, assume that \mathcal{M} is large in \mathcal{L} , set $R = S \cap \mathcal{M}$, set $\overline{\mathcal{M}} = \mathcal{M}/Z$, and let \mathcal{N} be the pre-image in \mathcal{L} of $\overline{\mathcal{M}}^\perp$. Then $\mathcal{N} \trianglelefteq \mathcal{L}$, and $[\mathcal{N}, R] \leq Z$. As $S \cap F^*(\mathcal{L}) \in \Delta$ and $F^*(\mathcal{L}) \leq \mathcal{M}$ we have $R \in \Delta$, and thus \mathcal{N} is a subgroup of the group $N_{\mathcal{L}}(R)$. Then $[O^p(\mathcal{N}), R] = 1$. By 6.3 and 6.10 we have $\Sigma_R(\mathcal{F}) \subseteq \Delta$, so $O^p(\mathcal{N}) \leq \mathcal{M}^\perp$ by 6.5. Thus $S \cap O^p(\mathcal{N}) \leq Z(\mathcal{N})$, and $O^p(\mathcal{N})$ is a p' -group, normal in $N_{\mathcal{L}}(R)$. As $N_{\mathcal{L}}(R)$ is of characteristic p we conclude that \mathcal{N} is a p -group, so $\mathcal{N} \leq O_p(\mathcal{L})$, and thus $\mathcal{N} \leq \mathcal{M}$. This shows that $\overline{\mathcal{M}}$ is large in $\overline{\mathcal{L}}$, and completes the proof of (5).

It is immediate from (5) that $F^*(\mathcal{L})/Z = F^*(\mathcal{L}/Z)$. Then (4) yields $\overline{\Delta} = \delta(\overline{\mathcal{F}})$ and completes the proof. \square

Lemma 7.3. *Let $P \leq S$ be a subgroup of S , and suppose that $O_p(\mathcal{L})P \in \delta(\mathcal{F})$. Then $P \in \delta(\mathcal{F})$.*

Proof. Set $\mathcal{M} = F^*(\mathcal{L})$, $R = S \cap \mathcal{M}$, and $\mathcal{D} = \mathcal{F}_R(\mathcal{M})$. As $O_p(\mathcal{L})P \in \delta(\mathcal{F})$ we have $R \cap O_p(\mathcal{L})P \in \mathcal{D}^s$. As $O_p(\mathcal{L}) \leq R$ we have $R \cap O_p(\mathcal{L})P = O_p(\mathcal{L})(R \cap P)$, and then $R \cap P \in \mathcal{D}^s$ by II.6.2(c). That is, $P \in \delta(\mathcal{F})$. \square

For the remainder of this section (\mathcal{L}, Δ, S) will be a regular locality on \mathcal{F} . The main goal is to show that every partial normal subgroup of \mathcal{L} is itself a regular locality.

Lemma 7.4. *Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} , and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} such that $O_p(\mathcal{L})C_S(\mathcal{N}) \leq \mathcal{N}$. Adopt the notation of 2.1, and set $\Gamma = \{U \leq T \mid U \in \Delta\}$. Then (\mathcal{N}, Γ, T) is a regular locality on \mathcal{E} , and $F^*(\mathcal{N}) = F^*(\mathcal{L})$.*

Proof. The hypothesis that $O_p(\mathcal{L})C_S(\mathcal{N})$ be contained in \mathcal{N} is equivalent to $F^*(\mathcal{L}) \leq \mathcal{N}$. Then Δ is the overgroup closure in S of Γ , and so $S_w \cap T \in \Gamma$ for each $w \in \mathbf{W}(\mathcal{N}) \cap \mathbf{D}$. Then (\mathcal{N}, Γ, T) is a locality by 1.2, and indeed a locality on \mathcal{E} by the definition \mathcal{E} as $\mathcal{F}_T(\mathcal{N})$. Here $\mathcal{E}^{cr} \subseteq \partial(\mathcal{F})$, so $\mathcal{E}^{cr} \subseteq \Delta$ by 6.10; and \mathcal{N} is then a proper locality by II.2.6(a). We note that $N_{\mathcal{L}}(T)$ acts on \mathcal{L} (and hence on \mathcal{N} , and also on \mathcal{E}) by conjugation, since $P \cap T \in \Delta$ for all $P \in \Delta$. Then $O_p(\mathcal{N}) = O_p(\mathcal{L})$ by the Frattini Lemma (I.3.12).

Set $\Gamma^+ = \mathcal{E}^s$, and let Δ^+ be the overgroup closure of Γ^+ in S . As $T \in \mathcal{F}^q$ by 4.5(a), we have $\Gamma^+ \subseteq \mathcal{F}^s$ by 4.3(c), and then $\Delta^+ \subseteq \mathcal{F}^s$ by II.6.2(a). As $N_{\mathcal{L}}(T)$ acts on \mathcal{E} , it follows that Δ^+ is \mathcal{F} -closed. Let $(\mathcal{L}^+, \Delta^+, S)$ be the expansion of \mathcal{L} to a proper locality on \mathcal{F} , via Theorem II.A1. For any $\mathcal{K} \trianglelefteq \mathcal{L}$ let $\mathcal{K}^+ \trianglelefteq \mathcal{L}^+$ be the partial normal subgroup of \mathcal{L}^+ , given by Theorem II.A2, whose intersection with \mathcal{L} is \mathcal{K} . Notice that $(\mathcal{N}^+, \Gamma^+, T)$ is a proper locality on \mathcal{E} , and so \mathcal{N}^+ is an expansion of \mathcal{N} . As $N_{\mathcal{L}}(T) = N_{\mathcal{L}^+}(T)$ permutes the set $\mathbb{M}(\mathcal{N}^+)$ of large partial normal subgroups \mathcal{K} of \mathcal{N}^+ containing $O_p(\mathcal{N})$, it follows from the Frattini Lemma that $F^*(\mathcal{N}^+) \trianglelefteq \mathcal{L}^+$.

Set $\mathcal{K}^+ = F^*(\mathcal{N}^+)$. Applying 6.4 with \mathcal{N}^+ and \mathcal{K}^+ in the roles of \mathcal{L} and \mathcal{N} , we obtain

$T \cap (\mathcal{K}^+)^\perp \leq Z(\mathcal{K}^+)$. Then 6.5 yields

$$O_{\mathcal{L}^+}^p((\mathcal{K}^+)^\perp) \leq (\mathcal{N}^+)^\perp.$$

But $(\mathcal{N}^+)^\perp = (\mathcal{N}^\perp)^+$ by 5.7(d), and $\mathcal{N}^\perp = Z(\mathcal{N})$ by 6.4. Thus $(\mathcal{K}^+)^\perp$ is a normal p -subgroup of \mathcal{L}^+ , and so $(\mathcal{K}^+)^\perp \leq O_p(\mathcal{L}^+)$. As $O_p(\mathcal{L}^+) = O_p(\mathcal{N}^+) = O_p(\mathcal{K}^+)$, we conclude that \mathcal{K}^+ is a large partial normal subgroup of \mathcal{L}^+ , and hence $F^*(\mathcal{L}^+) \leq \mathcal{K}^+$. The reverse inclusion holds since $F^*(\mathcal{L})^+$ is a large partial normal subgroup of \mathcal{N}^+ containing $O_p(\mathcal{N}^+)$, and since \mathcal{K}^+ is by definition the intersection of all such. Moreover, Theorem II.A2 then yields:

$$\mathcal{K}^+ \cap \mathcal{L} = F^*(\mathcal{L})^+ \cap \mathcal{L} = F^*(\mathcal{L}),$$

which in turn yields $F^*(\mathcal{E}) = \mathcal{F}^*(\mathcal{F})$. Then $\Gamma = \delta(\mathcal{E})$, and the proof is complete. \square

Lemma 7.5. *Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} , and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} such that $\mathcal{L} = O_p(\mathcal{L})\mathcal{N}$. Adopt the notation of 2.1, and set $\Gamma = \{U \leq T \mid U \in \Delta\}$. Then (\mathcal{N}, Γ, T) is a regular locality on \mathcal{E} , and $F^*(\mathcal{N}) = F^*(\mathcal{L}) \cap \mathcal{N}$.*

Proof. Set $\mathcal{K} = F^*(\mathcal{L}) \cap \mathcal{N}$. Then

$$F^*(\mathcal{L}) = F^*(\mathcal{L}) \cap \mathcal{L} = F^*(\mathcal{L}) \cap O_p(\mathcal{L})\mathcal{N} = O_p(\mathcal{L})\mathcal{K},$$

by the Dedekind Lemma. Then 7.3 implies that $P \cap T \in \Gamma$ for each $P \in \Delta$ such that $O_p(\mathcal{L}) \leq P$. Then $S_w \cap T \in \Gamma$ for each $w \in \mathbf{W}(\mathcal{N}) \cap \mathbf{D}$, and so (\mathcal{N}, Γ, T) is a locality on \mathcal{E} .

As in the proof of 7.4, it will be necessary to consider “versions” of \mathcal{L} and of \mathcal{N} with sets of objects other than Δ and Γ . Let Δ_0 be the overgroup closure of Γ in S . Thus $P \in \Delta_0$ if and only if $P \cap T \in \Gamma$, so $O_p(\mathcal{L})\Delta \subseteq \Delta_0 \subseteq \Delta$, and the restriction of \mathcal{L} to Δ_0 is then equal (as a partial group) to \mathcal{L} . Set $\Gamma^+ = \mathcal{E}^s$. As $C_{\mathcal{L}}(S)$ is a p -group, we have $O^p(C_{\mathcal{L}}(T)) = 1$, and so $T \in \mathcal{F}^q$ by II.2.8(c). Then $\Gamma^+ \subseteq \mathcal{F}^s$ by 4.3(c). Let Δ_0^+ be the overgroup closure of Γ^+ in S . Then Δ_0^+ is \mathcal{F} -closed, and $\Delta_0 \subseteq \Delta_0^+ \subseteq \mathcal{F}^s$. Let $(\mathcal{L}^+, \Delta_0^+, S)$ be the Δ_0^+ -expansion of \mathcal{L} to a proper locality on \mathcal{F} , and for each partial normal subgroup \mathcal{H} of \mathcal{L} let \mathcal{H}^+ be the partial normal subgroup of \mathcal{L}^+ such that $\mathcal{H} = \mathcal{H}^+ \cap \mathcal{L}$. As $O_p(\mathcal{L})\mathcal{N}^+ \cap \mathcal{L} = \mathcal{L}$ we then have $\mathcal{L}^+ = O_p(\mathcal{L})\mathcal{N}^+$.

Let $P \in \mathcal{E}^{cr}$ and set $Q = O_p(\mathcal{L})P$. Then $Q \in \Delta_0^+$, and

$$N_{\mathcal{L}^+}(Q) = O_p(\mathcal{L})N_{\mathcal{N}^+}(Q) = O_p(\mathcal{L})N_{\mathcal{N}^+}(P).$$

Since $P \in \mathcal{E}^{cr}$ we have $P = O_p(N_{\mathcal{N}^+}(P))$, and then $Q = O_p(N_{\mathcal{L}^+}(Q))$. Then $Q \in \mathcal{F}^{cr}$ by II.2.8(a), so $Q \in \Delta_0$. Then $P = Q \cap T \in \Gamma$, and thus $\mathcal{E}^{cr} \subseteq \Gamma$. We have thus shown that (\mathcal{N}, Γ, T) is proper.

Set $\mathcal{H} = \mathcal{K}^\perp \cap \mathcal{N}$. As $O_p(\mathcal{N}) \leq \mathcal{K}$, 5.7(c) yields $[O_p(\mathcal{N}), \mathcal{H}] = 1$. As $[O_p(\mathcal{L}), \mathcal{H}] \leq O_p(\mathcal{N})$ it follows from II.2.6(c) that $[O_p(\mathcal{L}), O_{\mathcal{N}}^p(\mathcal{H})] = 1$. Thus $T \cap O_{\mathcal{N}}^p(\mathcal{H}) \leq C_S(F^*(\mathcal{L}))$, and 6.4 then yields

$$T \cap O_{\mathcal{N}}^p(\mathcal{H}) \leq Z(O_{\mathcal{N}}^p(\mathcal{H})).$$

Then $O_{\mathcal{N}}^p(\mathcal{H})$ is a p -group by 3.17, and so $\mathcal{H} \leq T$. As $\mathcal{H} \trianglelefteq \mathcal{L}$ we get $\mathcal{H} \leq O_p(\mathcal{L})$. Then $\mathcal{H} \leq \mathcal{K}$, and since $C_T(\mathcal{K}) \leq \mathcal{H}$ we conclude that $O_p(\mathcal{N})C_T(\mathcal{K}) \leq \mathcal{K}$. Thus $F^*(\mathcal{N}) \leq \mathcal{K}$, and $O_p(\mathcal{L})F^*(\mathcal{N}) \leq F^*(\mathcal{L})$.

In order to obtain the reverse inclusion, observe first of all that

$$C_S(O_p(\mathcal{L})F^*(\mathcal{N}^+)) \leq O_p(\mathcal{L})C_S(F^*(\mathcal{N})^+) = O_p(\mathcal{L})C_T(F^*(\mathcal{N}^+)) \leq O_p(\mathcal{L}).$$

This shows:

$$(*) \quad F^*(\mathcal{L}^+) \leq O_p(\mathcal{L})F^*(\mathcal{N}^+).$$

Since $F^*(\mathcal{N}^+) = F^*(\mathcal{N})^+$ by 6.9, and since

$$O_p(\mathcal{L})F^*(\mathcal{N})^+ \cap \mathcal{L} = O_p(\mathcal{L})(F^*(\mathcal{N})^+ \cap \mathcal{L}) = O_p(\mathcal{L})F^*(\mathcal{N}),$$

we get

$$(O_p(\mathcal{L})F^*(\mathcal{N}))^+ = O_p(\mathcal{L})F^*(\mathcal{N})^+ = O_p(\mathcal{L})F^*(\mathcal{N}^+).$$

Then $(*)$ yields $F^*(\mathcal{L}) \leq O_p(\mathcal{L})F^*(\mathcal{N})$, and thus $F^*(\mathcal{L}) = O_p(\mathcal{L})F^*(\mathcal{N})$.

We have still to show that (\mathcal{N}, Γ, T) is a regular locality. Thus, it remains to show that $\Gamma = \delta(\mathcal{E})$. Let $X \in \delta(\mathcal{E})$ and set $U = X \cap F^*(\mathcal{N})$. As $\mathcal{E} = \mathcal{E}^+$ by 3.14, the definition of $\delta(\mathcal{E})$ yields $U \in F^*(\mathcal{E})^s$. Here $T \cap F^*(\mathcal{E}) \in \mathcal{E}^q$ by 4.5(a), so $U \in \mathcal{E}^s$ by 4.3(c). Then $U \in \mathcal{F}^s$ since we have already seen that $\mathcal{E}^s \subseteq \mathcal{F}^s$. As $U \leq F^*(\mathcal{N}) \leq F^*(\mathcal{L})$ it follows that $U \in F^*(\mathcal{F})^s$. Then $U \in \delta(\mathcal{F}) = \Delta$ and $X \in \Delta$. Thus $X \in \Gamma$, and we have $\delta(\mathcal{E}) \subseteq \Gamma$. Now let $Y \in \Gamma$ and set $V = Y \cap F^*(\mathcal{L})$. Then $V \in \Gamma$ since $\Delta = \delta(\mathcal{F})$. Then $N_{\mathcal{L}}(V)$ is of characteristic p , so also $N_{F^*(\mathcal{N})}(V)$ is of characteristic p by II.2.6(a). As $V \leq F^*(\mathcal{L}) \cap \mathcal{N} = F^*(\mathcal{N})$ we conclude that $V \in F^*(\mathcal{E})^s$, so $V \in \delta(\mathcal{E})$. Then $Y \in \delta(\mathcal{E})$, and thus $\Gamma = \delta(\mathcal{E})$, as required. \square

Lemma 7.6. *Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} , and assume the “product setup” of 2.9, with $\mathcal{L} = \mathcal{N}_1\mathcal{N}_2$. Then:*

- (a) $\mathcal{N}_i \leq (\mathcal{N}_{3-i})^\perp$ for $i = 1$ and 2 .
- (b) Each $(\mathcal{N}_i, \delta(\mathcal{E}_i), T_i)$ is a regular locality on \mathcal{E}_i , and $\delta(\mathcal{E}_i) = \{U \leq T_i \mid UC_S(\mathcal{N}_i) \in \Delta\}$.
- (c) $F^*(\mathcal{L}) = F^*(\mathcal{N}_1)F^*(\mathcal{N}_2)$.

Proof. We have $\mathcal{N}_2 \leq C_{\mathcal{L}}(T_1)$ by 2.9. Applying II.7.4 to the locality $N_{\mathcal{L}}(T_1)$ then yields

$$(1) \quad O^p N_{\mathcal{L}}(T_1)(\mathcal{N}_2) \leq O_{N_{\mathcal{L}}(T_1)}^p(C_{\mathcal{L}}(T_1)) \leq (\mathcal{N}_1)^\perp.$$

By definition, $O_{N_{\mathcal{L}}(T_1)}^p(C_{\mathcal{L}}(T_1))$ is the intersection of the set \mathbb{K} of partial normal subgroups \mathcal{K} of $N_{\mathcal{L}}(T_1)$ such that:

- (2) $\mathcal{K} \leq \mathcal{N}_2$ and $\mathcal{K}T_2 = \mathcal{N}_2$.

Evidently \mathbb{K} contains the set of partial normal subgroups \mathcal{K} of \mathcal{L} such that (2) holds, so $O_{\mathcal{L}}^p(\mathcal{N}_2) \leq O^p N_{\mathcal{L}}(T_1)(\mathcal{N}_2)$. Then (1) yields $O_{\mathcal{L}}^p(\mathcal{N}_2) \leq (\mathcal{N}_1)^\perp$. Since $\Sigma_{T_1}(\mathcal{F}) \subseteq \Delta$ we have $C_S(\mathcal{N}_1) = (\mathcal{N}_1)^\perp$ by 5.7(b), and so $T_2 \leq (\mathcal{N}_1)^\perp$. Thus $\mathcal{N}_2 \leq (\mathcal{N}_1)^\perp$, and point (a) follows.

Set $\mathcal{M} = F^*(\mathcal{L})$, set $\mathcal{M}_1 = \mathcal{M} \cap \mathcal{N}_1$, and set $\mathcal{K} = (\mathcal{M}_1)^\perp \cap \mathcal{N}_1$. Then $\mathcal{K} \trianglelefteq \mathcal{L}$ by 5.7(a), and

$$\mathcal{K} \cap \mathcal{M} = \mathcal{K} \cap \mathcal{M}_1 \leq Z(\mathcal{K}),$$

so 6.5 yields $O_{\mathcal{L}}^p(\mathcal{K}) \leq \mathcal{M}^\perp$. Now 4.7 yields $\mathcal{M}^\perp = Z(\mathcal{M})$ and $O_{\mathcal{L}}^p(\mathcal{K}) = \mathbf{1}$. Thus \mathcal{K} is a normal p -subgroup of \mathcal{L} , and so $\mathcal{K} \leq O_p(\mathcal{L})$. Notice that since $[\mathcal{N}_i, T_{3-i}] = \mathbf{1}$ we have $O_p(\mathcal{L}) = O_p(\mathcal{N}_1)O_p(\mathcal{N}_2)$. As $O_p(\mathcal{L}) \leq \mathcal{M}$ we then have $O_p(\mathcal{L}) = O_p(\mathcal{M}_1)O_p(\mathcal{M}_2)$, and $\mathcal{K} \leq O_p(\mathcal{M}_1)$. We now compute:

$$\begin{aligned} C_{T_1}(\mathcal{M}_1) &\leq C_S(\mathcal{M}_1\mathcal{M}_2) = C_S(\mathcal{M}_1) \cap C_S(\mathcal{M}_2) \\ &= C_{T_1}(\mathcal{M}_1)T_2 \cap C_{T_2}(\mathcal{M}_2)T_1 = Z(\mathcal{M}_1)T_2 \cap Z(\mathcal{M}_2)T_1 \\ &= Z(\mathcal{M}_1)(T_2 \cap Z(\mathcal{M}_2)T_1) = Z(\mathcal{M}_1)Z(\mathcal{M}_2)(T_1 \cap T_2) \leq O_p(\mathcal{L}) \\ &\leq \mathcal{M}_1\mathcal{M}_2. \end{aligned}$$

This shows that $C_{T_1}(\mathcal{M}_1) \leq \mathcal{M}_1$, and that $\mathcal{M}_1\mathcal{M}_2$ is a large partial normal subgroup of \mathcal{L} containing $O_p(\mathcal{L})$. We have shown:

- (3) $F^*(\mathcal{L}) = \mathcal{M}_1\mathcal{M}_2$, and
- (4) $O_p(\mathcal{N}_i)C_S(\mathcal{M}_i) \leq \mathcal{M}_i$ ($i = 1, 2$).

Next, set $\Gamma_i = \{PT_{3-i} \cap T_i \mid P \in \Delta\}$ and set $\Delta_0 = \Gamma_1\Gamma_2$. Then $(\mathcal{N}_i, \Gamma_i, T_i)$ is a proper locality on \mathcal{E}_i , Δ_0 is an \mathcal{F} -closed subset of Δ containing \mathcal{F}^{cr} , and the restriction of caL to Δ_0 is equal to \mathcal{L} as a partial group, by 2.11 and by 2.10(a). Set $\Gamma_i^+ = \mathcal{E}_i^s$ and let Δ_0^+ be the overgroup closure of $\Gamma_1^+\Gamma_2^+$. Then Δ_0^+ is an \mathcal{F} -closed subset of \mathcal{F}^s by 2.10(e). Form the corresponding expansions $(\mathcal{N}_i)^+$ and \mathcal{L}^+ via Theorem II.A1. Let $\mathcal{K}_i^+ = F^*(\mathcal{N}_i)^+$ be the partial normal subgroup of $(\mathcal{N}_i)^+$ corresponding to $\mathcal{K}_i = F^*(\mathcal{N}_i)$ via Theorem II.A2. Note that $\mathcal{N}_i^+ \leq (\mathcal{N}_{3-i}^+)^\perp$ (a) and 5.7(d). Then 5.7(c) implies that $\mathcal{K}_i \trianglelefteq \mathcal{L}$ and $\mathcal{K}_i^+ \trianglelefteq \mathcal{L}^+$. Let $(\mathcal{K}_1\mathcal{K}_2)^+$ be the partial normal subgroup of \mathcal{L}^+ whose intersection with \mathcal{L} is $\mathcal{K}_1\mathcal{K}_2$. Then $(\mathcal{K}_1\mathcal{K}_2)^+ = \mathcal{K}_1^+\mathcal{K}_2^+$ by II.5.3. We compute:

$$\begin{aligned} C_S(\mathcal{K}_1^+\mathcal{K}_2^+) &= C_S(\mathcal{K}_1^+) \cap C_S(\mathcal{K}_2^+) = C_{T_1}(\mathcal{K}_1^+)T_2 \cap C_{T_2}(\mathcal{K}_2^+)T_1 \\ &= Z(\mathcal{K}_1^+)T_2 \cap Z(\mathcal{K}_2^+)T_1 = Z(\mathcal{K}_1^+)Z(\mathcal{K}_2^+)(T_1 \cap T_2) \leq \mathcal{K}_1^+\mathcal{K}_2^+, \end{aligned}$$

and so $(\mathcal{K}_1\mathcal{K}_2)^+$ is a large partial normal subgroup of \mathcal{L}^+ . As also $O_p(\mathcal{F}) = O_p(\mathcal{L}) \leq \mathcal{K}_1\mathcal{K}_2$, we conclude that $F^*(\mathcal{L}^+) \leq (\mathcal{K}_1\mathcal{K}_2)^+$. This yields $\mathcal{M} \leq \mathcal{K}_1\mathcal{K}_2$. Since also $\mathcal{K}_i \leq \mathcal{M}_i$ by (4), we obtain $\mathcal{K}_i = \mathcal{M}_i$. That is, (c) holds.

Let $U_i \in F^*(\mathcal{E}_i)^s$. Then $U_i \in \delta(\mathcal{E}_i) \subseteq (\mathcal{E}_i)^s$, and then $U_1U_2 \in \mathcal{F}^s$ by 2.10(e). As $U_1U_2 \leq F^*(\mathcal{L})$ by (c), we obtain $\delta(\mathcal{E}_1)\delta(\mathcal{E}_2) \subseteq \delta(\mathcal{F})$. As $\delta(\mathcal{F}) = \Delta$ we have $U_1(T_2 \cap \mathcal{M}_2) \in \Delta$ in particular, and so $U_1(T_1 \cap T_2) \in \Gamma_1$. This shows that $Z(\mathcal{N}_1)\delta(\mathcal{E}_1) \subseteq \Gamma_1$.

Let $Q_1 \in \Gamma_1$. Then $Q_1 = P \cap T_1$ for some $P \in \Delta$ with $T_2 \leq P$. Then $P \cap \mathcal{M} \in F^*(\mathcal{F})^s$ as $\Delta = \delta(\mathcal{F})$. Here $P \cap \mathcal{M} = (P \cap \mathcal{M}_1)(T_2 \cap \mathcal{M}_2)$, so $P \cap \mathcal{M}_1 \in F^*(\mathcal{E}_1)^s$ by 2.10(e). Thus

$Q_1 \in \delta(\mathcal{E}_1)$, and this shows that $\Gamma_1 \subseteq \delta(\mathcal{E}_1)$. Since $Z(\mathcal{N}_1) \leq S_w$ for all $w \in \mathbf{W}(\mathcal{N}_1)$, the expansion of $(\mathcal{N}_1, \Gamma_1, T_1)$ to a regular locality on $\delta(\mathcal{E}_1)$ has the same underlying partial group \mathcal{N}_1 . This completes the proof of (b), and of the lemma. \square

Theorem 7.7. *Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} , and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup. Set $T = S \cap \mathcal{N}$, $\mathcal{E} = \mathcal{F}_T(\mathcal{N})$, and $\Gamma = \{U \leq T \mid C_S(\mathcal{N})U \in \Delta\}$. Then:*

- (a) (\mathcal{N}, Γ, T) is a regular locality on \mathcal{E} .
- (b) $F^*(\mathcal{N}) = \mathcal{N} \cap F^*(\mathcal{L})$.
- (c) $\mathcal{N}^\perp = C_{\mathcal{L}}(\mathcal{N})$. That is, \mathcal{N}^\perp is the set of elements $g \in \mathcal{L}$ such that $x^g = x$ for all $x \in \mathcal{N}$.
- (d) $O_p(\mathcal{E}) = O_p(\mathcal{N}) \trianglelefteq \mathcal{L}$.
- (e) $(g^{-1}, x, g) \in \mathbf{D}$ for all $x \in \mathcal{N}$ and all $g \in N_{\mathcal{L}}(T)$. Moreover, the mapping $c_g : \mathcal{N} \rightarrow \mathcal{N}$ given by $x \mapsto \Pi(g^{-1}, x, g)$ is an automorphism of \mathcal{N} , and the mapping $\eta : N_{\mathcal{L}}(T) \rightarrow \text{Aut}(\mathcal{N})$ given by $g \mapsto c_g$ is a homomorphism of partial groups with kernel \mathcal{N}^\perp .
- (f) The image of the natural homomorphism $N_{\mathcal{L}}(T) \rightarrow \text{Aut}(T)$ is contained in $\text{Aut}(\mathcal{E})$.

Proof. If \mathcal{N} is large and $O_p(\mathcal{L}) \leq \mathcal{N}$ then (a) and (b) are given by 7.3 and 7.4. Set $\mathcal{M} = \mathcal{N}\mathcal{N}^\perp$. Thus (a) and (b) hold with $O_p(\mathcal{L})\mathcal{M}$ in the role of \mathcal{N} . In order to prove (a) and (b) in general, we may then assume that $\mathcal{L} = O_p(\mathcal{L})\mathcal{M}$. By 7.5 we then obtain (a) and (b) for \mathcal{M} in the role of \mathcal{N} . In this way we reduce to the case where $\mathcal{L} = \mathcal{M}$, where (a) and (b) are then given by 7.6.

In proving (c) we may now assume that \mathcal{L} is the regular locality $\mathcal{N}\mathcal{N}^\perp$. Let $f \in \mathcal{N}$ and $g \in \mathcal{N}^\perp$, set $U = S_f \cap T$, and set $V = S_g \cap T^\perp$. Then $UV \in \Delta$ by 7.6(b), and so $(f, g) \in \mathbf{D}$. Then $[f, g] = \mathbf{1}$ by 5.7(c), and this yields point (c). That $O_p(\mathcal{E}) = O_p(\mathcal{N})$ is now a consequence of (a) and II.2.3. Point (d) will then follow from (f) and the factorization $\mathcal{L} = N_{\mathcal{L}}(T)\mathcal{N}$ given by the Frattini Lemma. Thus, it remains to prove (e) and (f).

Set $H = N_{\mathcal{L}}(TT^\perp)$ and set $\mathcal{L}_T = N_{\mathcal{L}}(T)$. Then $(\mathcal{L}_T, \Delta, S)$ is a locality, and $\mathcal{N}^\perp \trianglelefteq \mathcal{L}_T$, so the Frattini Lemma yields $\mathcal{L}_T = \mathcal{N}^\perp H$. Here $TT^\perp \in \Delta$ by an application of (a) to $\mathcal{N}\mathcal{N}^\perp$ in the role of \mathcal{N} , so H is a subgroup of \mathcal{L} . Let $x \in \mathcal{N}$ and set $P = S_x \cap T$. Then $P \in \Gamma$. Now let $g \in \mathcal{L}_T$ and employ the Splitting Lemma so as to write $g = yh$ with $y \in \mathcal{N}^\perp$ and $h \in H$, and with $S_g = S_{(y,h)}$. Set $Q = S_y \cap T^\perp$. Then $PQ \in \Delta$ by 2.11, and one may verify that the word $(h^{-1}, y^{-1}, x, y, h)$ is in \mathbf{D} via $(PQ)^h$. Thus $(g^{-1}, x, g) \in \mathbf{D}$, and (f) follows from 2.3. A similar argument shows that if $(x_1, \dots, x_n) \in \mathbf{D}(\mathcal{N})$ then

$$(g^{-1}, x_1, g, \dots, g^{-1}, x_n, g) \in \mathbf{D},$$

and hence that the conjugation map $c_g : \mathcal{N} \rightarrow \mathcal{N}$ is an endomorphism of \mathcal{N} as a partial group. One checks that $c_g \circ c_{g^{-1}}$ is the identity map on \mathcal{N} , so we have a mapping $\eta : \mathcal{L}_T \rightarrow \text{Aut}(\mathcal{N})$ given by $g \mapsto c_g$. The restriction of η to H is easily seen to be a homomorphism (of groups), and a further exercise with the Splitting Lemma will verify that η itself is a homomorphism of partial groups, completing the proof of (e). \square

Definition 7.8. A partial subgroup \mathcal{K} of a locality \mathcal{L} is *subnormal* in \mathcal{L} (denoted $\mathcal{K} \trianglelefteq \mathcal{L}$) if there exists a sequence $(\mathcal{N}_0, \dots, \mathcal{N}_k)$ of partial subgroups of \mathcal{L} such that $\mathcal{K} = \mathcal{N}_0 \trianglelefteq \mathcal{N}_1 \trianglelefteq \dots \trianglelefteq \mathcal{N}_k = \mathcal{L}$.

Corollary 7.9. Let \mathcal{L} be a regular locality and let $\mathcal{K} \trianglelefteq \mathcal{L}$ be a partial subnormal subgroup of \mathcal{L} . Then \mathcal{K} is a regular locality.

Proof. By induction on the length of a subnormal chain from \mathcal{K} to \mathcal{L} it suffices to show that \mathcal{K} is regular in the case that $\mathcal{K} \trianglelefteq \mathcal{L}$, in which case we are done by 7.7(a). \square

Lemma 7.10. Let \mathcal{L} be a regular locality, let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup, and let \mathbb{K} be a non-empty set of partial normal subgroups of \mathcal{N} . Set $T = S \cap \mathcal{N}$, and let $\Lambda = N_{\text{Aut}(\mathcal{N})}(T)$ be the set of automorphisms of \mathcal{N} (as a partial group) which leave T invariant. Suppose that \mathbb{K} is invariant under Λ , and set $\mathcal{K} = \bigcap \mathbb{K}$. Then $\mathcal{K} \trianglelefteq \mathcal{L}$.

Proof. We have $\mathcal{K} \trianglelefteq \mathcal{N}\mathcal{N}^\perp$ by 7.7(c), and \mathcal{K} is S -invariant since, by I.2.9, S acts on \mathcal{L} by conjugation. Thus $\mathcal{K} \trianglelefteq \mathcal{M} := O_p(\mathcal{L})\mathcal{N}\mathcal{N}^\perp$. Set $H = N_{\mathcal{L}}(S \cap \mathcal{M})$. Then H is a subgroup of $N_{\mathcal{L}}(S \cap F^*(\mathcal{L}))$. As \mathcal{L} is regular we have $\mathbf{D} = \{w \in \mathbf{W}(\mathcal{L}) \mid S_w \cap F^*(\mathcal{L}) \in \Delta\}$, so H acts on \mathcal{L} by conjugation. Then $O^p(\mathcal{N})$ is H -invariant, and we may employ I.3.13 to conclude that $\mathcal{K} \trianglelefteq \mathcal{L}$. \square

Recall from II.7.1 that $O^p(\mathcal{L})$ is defined to be $O_{\mathcal{L}}^p(\mathcal{L})$ (and similarly for $O^{p'}(\mathcal{L})$). Recall also the definition of $[\mathcal{L}, \mathcal{L}]$ from 3.16.

Corollary 7.11. Let \mathcal{L} be a regular locality and let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup. Then $O^p(\mathcal{N})$, $O^{p'}(\mathcal{N})$, and $[\mathcal{N}, \mathcal{N}]$ are partial normal subgroups of \mathcal{L} . Moreover, we have $O_{\mathcal{L}}^p(\mathcal{N}) = O^p(\mathcal{N})$ and $O_{\mathcal{L}}^{p'}(\mathcal{N}) = O^{p'}(\mathcal{N})$.

Proof. Let \mathbb{K} be the set of partial normal subgroups of \mathcal{N} defined by any one of the following three conditions: (1) $\mathcal{H} \in \mathbb{K}$ if $\mathcal{H}T = \mathcal{N}$, (2) $\mathcal{H} \in \mathbb{K}$ if $T \leq \mathcal{H}$, and (3) $\mathcal{H} \in \mathbb{K}$ if \mathcal{N}/\mathcal{H} is an abelian group. Set $\mathcal{K} = \bigcap \mathbb{K}$. Then $\mathcal{K} \trianglelefteq \mathcal{L}$ by 7.10, and \mathcal{K} is variously $O^p(\mathcal{N})$, $O^{p'}(\mathcal{N})$, or $[\mathcal{N}, \mathcal{N}]$. Now let \mathbb{K}' be the set of all partial normal subgroups \mathcal{H}' of \mathcal{L} such that $\mathcal{H}'T = \mathcal{N}$. Then $O_{\mathcal{L}}^p(\mathcal{N}) = \bigcap \mathbb{K}'$ by definition, and $O_{\mathcal{L}}^p(\mathcal{N}) \leq O^p(\mathcal{N})$ since $O^p(\mathcal{N}) \in \mathbb{K}'$. The reverse inclusion holds since $O_{\mathcal{L}}^p(\mathcal{N}) \in \mathbb{K}$ (with \mathbb{K} as in (1)). Thus $O_{\mathcal{L}}^p(\mathcal{N}) = O^p(\mathcal{N})$, and similarly $O_{\mathcal{L}}^{p'}(\mathcal{N}) = O^{p'}(\mathcal{N})$. \square

Section 8: Components

Throughout this section, (\mathcal{L}, Δ, S) is a regular locality on \mathcal{F} . That is, \mathcal{L} is a proper locality on \mathcal{F} , and $\Delta = \delta(\mathcal{F})$. By 7.9, every partial subnormal subgroup \mathcal{K} of \mathcal{L} is then a regular locality, and $O^p(\mathcal{K}) \trianglelefteq \mathcal{L}$ by 7.11. We may write $\delta(\mathcal{K})$ for $\delta(\mathcal{F})$, and thus write also $(\mathcal{K}, \delta(\mathcal{K}), S \cap \mathcal{K})$ for the regular locality \mathcal{K} . A locality \mathcal{L} is *simple* if \mathcal{L} has exactly two partial normal subgroups.

Definition 8.1. A partial subnormal subgroup $\mathcal{K} \trianglelefteq \mathcal{L}$ is a *component* of \mathcal{L} if $\mathcal{K} = O^p(\mathcal{K})$ and $\mathcal{K}/Z(\mathcal{K})$ is simple.

Write $\text{Comp}(\mathcal{L})$ for the set of components of \mathcal{L} , and let

$$E(\mathcal{L}) = \langle \text{Comp}(\mathcal{L}) \rangle$$

be the partial subgroup of \mathcal{L} generated by the union of the components of \mathcal{L} . Recall that for subsets X, Y of \mathcal{L} , the notation $[X, Y] = \mathbf{1}$ indicates that the commutators $[x, y]$ are defined and are equal to $\mathbf{1}$ for all $x \in X$ and all $y \in Y$.

Lemma 8.2. *Let $\mathcal{K} \in \text{Comp}(\mathcal{L})$ and let $\mathcal{N} \trianglelefteq \mathcal{K}$. Then $Z(\mathcal{K}) = O_p(\mathcal{K})$, and either $\mathcal{N} = \mathcal{K}$ or $\mathcal{N} \leq Z(\mathcal{K})$.*

Proof. As \mathcal{K} is itself a regular locality, we may as well assume that $\mathcal{K} = \mathcal{L}$. We have $Z(\mathcal{L}) \leq O_p(\mathcal{L})$ by 2.7. Now let $\mathcal{N} \trianglelefteq \mathcal{L}$ be given with $\mathcal{N} \not\leq Z(\mathcal{L})$. Here $Z(\mathcal{L})\mathcal{N} \trianglelefteq \mathcal{L}$ by I.5.1. As $\mathcal{L}/Z(\mathcal{L})$ is simple, the Correspondence Theorem (I.4.8) yields $Z(\mathcal{L})\mathcal{N} = \mathcal{L}$.

Set $T = S \cap \mathcal{N}$. Then

$$O^p(\mathcal{N})S \geq Z(\mathcal{L})O^p(\mathcal{N})T = Z(\mathcal{L})\mathcal{N} = \mathcal{L}.$$

As $O^p(\mathcal{N}) \trianglelefteq \mathcal{L}$, by 7.11, we conclude that $O^p(\mathcal{L}) \leq O^p(\mathcal{N})$. Then $\mathcal{N} = \mathcal{L}$, since $\mathcal{L} = O^p(\mathcal{L})$. Thus, either $\mathcal{N} \leq Z(\mathcal{L})$ or $\mathcal{N} = \mathcal{L}$. The special case where $\mathcal{N} = O_p(\mathcal{L})$ then yields $O_p(\mathcal{L}) \leq Z(\mathcal{L})$, and completes the proof. \square

Lemma 8.3. *Let $\mathcal{K} \trianglelefteq \trianglelefteq \mathcal{L}$.*

- (a) *If $\mathcal{K} \leq \mathcal{N} \trianglelefteq \mathcal{L}$ then $\mathcal{K} \trianglelefteq \trianglelefteq \mathcal{N}$.*
- (b) *$O_p(\mathcal{K}) \leq O_p(\mathcal{L})$, and if $\mathcal{K} \trianglelefteq \mathcal{L}$ then $O_p(\mathcal{K}) \trianglelefteq \mathcal{L}$.*
- (c) *$O_p(\mathcal{L})\mathcal{K} \trianglelefteq \trianglelefteq \mathcal{L}$.*

Proof. Point (a) is immediate from the observation that the intersection of partial normal subgroups is again a partial normal subgroup.

In proving (b): suppose first that $\mathcal{K} \trianglelefteq \mathcal{L}$. Then $O_p(\mathcal{K}) \trianglelefteq \mathcal{L}$ by 7.11, with $\{O_p(\mathcal{K})\}$ in the role of \mathbb{K} . Now (b) follows by an obvious induction argument on the length of a subnormal chain from \mathcal{K} to \mathcal{L} .

Set $\overline{\mathcal{L}} = \mathcal{L}/O_p(\mathcal{L})$ and let $\overline{\mathcal{K}}$ be the image of \mathcal{K} in $\overline{\mathcal{L}}$ under the canonical projection $\rho : \mathcal{L} \rightarrow \overline{\mathcal{L}}$. The Correspondence Theorem (I.4.7) yields $\overline{\mathcal{K}} \trianglelefteq \trianglelefteq \overline{\mathcal{L}}$, and the preimage \mathcal{H} of $\overline{\mathcal{K}}$ is a partial subnormal subgroup of \mathcal{L} . This yields (c). \square

Lemma 8.4. *Let $\mathcal{K} \in \text{Comp}(\mathcal{L})$. Then:*

- (a) *$Z(\mathcal{K}) \leq O_p(\mathcal{L})$.*
- (b) *$[O_p(\mathcal{L}), \mathcal{K}] = \mathbf{1}$.*
- (c) *$\mathcal{K} = O^p(O_p(\mathcal{L})\mathcal{K})$.*

Proof. We have $Z(\mathcal{K}) \leq O_p(\mathcal{K})$ by 4.8, and then point (a) follows from 6.3(a).

Let X be a subgroup of \mathcal{K} of order prime to p . Then $O_p(\mathcal{L})X$ is a subgroup of \mathcal{L} . Set $Z_0 = O_p(\mathcal{L})$ and recursively define $Z_n := [Z_{n-1}, X]$ for $n \geq 1$. Then $Z_n \leq O_p(\mathcal{L})$ for all n , and since $\mathcal{K} \trianglelefteq \trianglelefteq \mathcal{L}$ we obtain $Z_n \leq \mathcal{K} \cap O_p(\mathcal{L})$ for n sufficiently large. Then $Z_{n+1} = 1$ since $O_p(\mathcal{K}) \leq Z(\mathcal{K})$; and then $Z_1 = 1$ by coprime action. In particular, we now have

$[O_p(\mathcal{L}), O^p(N_{\mathcal{K}}(P))] = 1$ for all $P \in \delta(\mathcal{K})$. One easily verifies (via I.2.9 for example) that $C_{\mathcal{L}}(O_p(\mathcal{L}))$ is a partial subgroup of \mathcal{L} , and thus

$$\langle O^p(N_{\mathcal{K}}(P)) \mid P \in \delta(\mathcal{K}) \rangle \leq C_{\mathcal{L}}(O_p(\mathcal{L})).$$

Then $O^p(\mathcal{L}) \leq C_{\mathcal{L}}(O_p(\mathcal{L}))$ by an application of 4.3, with \mathcal{K} in the role of \mathcal{L} . As $(x^{-1}, g^{-1}, x, g) \in \mathbf{D}$ for all $x \in O_p(\mathcal{L})$ and all $g \in \mathcal{L}$, we obtain (b).

As $O_p(\mathcal{L})\mathcal{K}$ is subnormal in \mathcal{L} by 6.3(b), $O_p(\mathcal{L})\mathcal{K}$ is a regular locality, and we may therefore take $\mathcal{L} = O_p(\mathcal{L})\mathcal{K}$ in proving (c). Then (b) yields $\mathcal{K} \trianglelefteq \mathcal{L}$. Now $O^p(\mathcal{L}) \leq \mathcal{K}$ as $\mathcal{K}S = \mathcal{L}$, and then $O^p(\mathcal{K}) \leq O^p(\mathcal{L})$ since

$$\mathcal{K} = \mathcal{L} \cap \mathcal{K} = O^p(\mathcal{L})S \cap \mathcal{K} = O^p(\mathcal{L})T.$$

The reverse inclusion $O^p(\mathcal{L}) \leq O^p(\mathcal{K})$ is given by

$$O^p(\mathcal{K})S = (O^p(\mathcal{K})T)S = \mathcal{K}S = \mathcal{L}.$$

□

Theorem 8.5. *Let (\mathcal{L}, Δ, S) be a regular locality which is not a group of characteristic p . Then $\text{Comp}(\mathcal{L}) \neq \emptyset$. Let $(\mathcal{K}_1, \dots, \mathcal{K}_m)$ be a non-redundant list of the components of \mathcal{L} , and let $E(\mathcal{L})$ be the partial subgroup of \mathcal{L} generated by $\bigcup \text{Comp}(\mathcal{L})$. Then*

$$F^*(\mathcal{L}) = O_p(\mathcal{L})E(\mathcal{L}) = O_p(\mathcal{L})\mathcal{K}_1 \cdots \mathcal{K}_m,$$

and

$$[\mathcal{K}_i, \mathcal{K}_j] = 1 \quad (1 \leq i \neq j \leq m).$$

Further, $E(\mathcal{L}) = O^p(F^*(\mathcal{L}))$, and $[O_p(\mathcal{L}), E(\mathcal{L})] = 1$.

Proof. If $O_p(\mathcal{L})$ is a large partial normal subgroup of \mathcal{L} then $O_p(\mathcal{L}) = F^*(\mathcal{L}) \in \Delta$, and then \mathcal{L} is a group of characteristic p , contrary to hypothesis. Thus $O_p(\mathcal{L})$ is not large. Set $\mathcal{H} = O_p(\mathcal{L})^\perp$, and let \mathbf{X} be the set of partial subnormal subgroups \mathcal{K} of \mathcal{H} such that \mathcal{K} is not a p -group. Regard \mathbf{X} as a poset via inclusion, and let \mathcal{K} be minimal in \mathbf{X} . Then $\mathcal{K} = O^p(\mathcal{K})$, and $\mathcal{K}/O_p(\mathcal{K})$ is simple. As $O_p(\mathcal{K}) \leq O_p(\mathcal{L})$ by 8.3(b), and since $\mathcal{K} \leq \mathcal{H} = C_{\mathcal{L}}(O_p(\mathcal{L}))$ by 7.7(c), we obtain $\mathcal{K} \in \text{Comp}(\mathcal{L})$. Here $\mathbf{X} \neq \emptyset$ as $\mathcal{H} \in \mathbf{X}$, so we have shown that $\text{Comp}(\mathcal{L}) \neq \emptyset$.

Set $\mathcal{M} = F^*(\mathcal{L})$ and let $\mathcal{K} \in \text{Comp}(\mathcal{L})$. Let $\mathcal{N} \trianglelefteq \mathcal{L}$ be a partial normal subgroup of \mathcal{L} which is minimal with respect to the condition $\mathcal{K} \trianglelefteq \mathcal{N}$. If $\mathcal{K} = \mathcal{L}$ then $\mathcal{M} = \mathcal{K}$ and $\{\mathcal{K}\} = \text{Comp}(\mathcal{L})$ by 8.2, and there is then nothing to prove. Thus we may assume that $\mathcal{K} \neq \mathcal{L}$. Then $\mathcal{N} \neq \mathcal{L}$, and induction on $|\mathcal{L}|$ then yields $\mathcal{K} \leq F^*(\mathcal{N})$. Then $\mathcal{K} \leq F^*(\mathcal{L})$ by 7.7(b). Thus $E(\mathcal{L}) \leq \mathcal{M}$, and we may therefore assume at this point that $\mathcal{L} = \mathcal{M}$.

With \mathcal{N} as above we have $\mathcal{L} = \mathcal{N}\mathcal{N}^\perp \neq \mathcal{N}$, and $\mathcal{L} \neq \mathcal{N}^\perp$. As $\mathcal{L} = F^*(\mathcal{L}) = F^*(\mathcal{N})F^*(\mathcal{N}^\perp)$, induction on $|S|$ yields $\mathcal{K} \trianglelefteq \mathcal{L} = E(\mathcal{L})$. Then $\mathcal{L} = \mathcal{K}\mathcal{K}^\perp$ is a product of pairwise commuting components, and it suffices now to show that \mathcal{K}^\perp contains each component \mathcal{K}' of \mathcal{L} other than \mathcal{K} .

Let $\mathcal{K}' \in \text{Comp}(\mathcal{L})$, and suppose that $\mathcal{K} \neq \mathcal{K}'$. Then $\mathcal{K} \cap \mathcal{K}' \leq Z(\mathcal{K})$ by 8.2. As $\mathcal{L}/Z(\mathcal{L})$ is regular by 7.2, we may assume $Z(\mathcal{L}) = \mathbf{1}$. Let \mathcal{K}' be a component with $\mathcal{K} \neq \mathcal{K}'$. Then $\mathcal{K}' \trianglelefteq \mathcal{L}$ and $[S \cap \mathcal{K}, S \cap \mathcal{K}'] = \mathbf{1}$. As $\mathcal{K}' = O^p(\mathcal{K}')$ we then have $\mathcal{K}' \leq O^p(C_{\mathcal{L}}(S \cap \mathcal{K}))$ by II.7.4, and then $\mathcal{K}' \leq \mathcal{K}^\perp$ as desired, by definition 5.5. □

Corollary 8.6. *Let (\mathcal{L}, Δ, S) be a regular locality and let \mathcal{N} be a partial normal subgroup of $E(\mathcal{L})$. Then $\mathcal{N} = E(\mathcal{N})Z$, where $Z \leq Z(E(\mathcal{L}))$ and where $E(\mathcal{N})$ is a product of components of \mathcal{L} .*

Proof. We may assume without loss of generality that $\mathcal{L} = E(\mathcal{L})$. Then $O_p(\mathcal{L}) = Z(\mathcal{L})$ by the final statement in 8.5. We have $\mathcal{N} = F^*(\mathcal{N})$ by 7.7(b), and thus $\mathcal{N} = O_p(\mathcal{N})E(\mathcal{N})$ where $E(\mathcal{N})$ is a product of components of \mathcal{N} . The components of \mathcal{N} are by definition components of \mathcal{L} , and $O_p(\mathcal{N}) \leq O_p(E(\mathcal{L}))$ by 7.7(d). Set $Z = O_p(\mathcal{N})$. Thus $Z \leq Z(\mathcal{L})$ and $\mathcal{N} = E(\mathcal{N})Z$. \square

Section 9: Im-partial subgroups and E -balance

Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} , and let $X \leq S$ be fully normalized in \mathcal{F} . We shall see that there is then a regular locality \mathcal{L}_X on $N_{\mathcal{F}}(X)$ which can be constructed in a somewhat indirect way from the partial group $N_{\mathcal{L}}(X)$. One of the goals of this section is to show that if $X \leq F^*(\mathcal{L})$ then \mathcal{L}_X can be produced directly as a subset of \mathcal{L} , and that the inclusion map $\mathcal{L}_X \rightarrow \mathcal{L}$ is a homomorphism of partial groups.

In the category of groups and homomorphisms one has the obvious equivalence between the notions of “subgroup of G ” and “image of a homomorphism into G ”. The situation is different in the category of partial groups, where partial subgroups are indeed images of homomorphisms, but where images of homomorphisms need not be partial subgroups. As an example, take \mathcal{L} to be the additive group of integers, and let \mathcal{H} be the subset $\{-1, 0, 1\}$ of \mathcal{L} . Then \mathcal{H} is the image of a homomorphism of partial groups (see I.1.2 and I.1.12), but \mathcal{H} is not a partial subgroup of \mathcal{L} (since partial subgroups of groups are subgroups).

Definition 9.1. Let \mathcal{H} and \mathcal{M} be partial groups. Then \mathcal{H} is an *im-partial subgroup* of \mathcal{M} if \mathcal{H} is the image in \mathcal{M} of a homomorphism of partial groups. Equivalently: $\mathbf{D}(\mathcal{H}) \subseteq \mathbf{D}(\mathcal{L})$ and $Pi_{\mathcal{H}}$ is the restriction of $\Pi_{\mathcal{M}}$ to $\mathbf{D}(\mathcal{H})$. Write $\mathcal{K} \preceq \mathcal{H}$ to indicate that \mathcal{K} is an im-partial subgroup of \mathcal{H} .

It is obvious that the relation \preceq is transitive. Of the long list (analogous to I.1.8) of properties of im-partial subgroups that one might compile, the following lemma provides the few instances that will be needed here.

Lemma 9.2. *Let \mathcal{M} be a partial group, let $\mathcal{H} \preceq \mathcal{M}$ be an im-partial subgroup, and let $\mathcal{K} \leq \mathcal{M}$ be a partial subgroup. Then:*

- (a) $\mathcal{H} \cap \mathcal{K}$ is an im-partial subgroup of \mathcal{K} and of \mathcal{M} .
- (b) $\mathcal{H} \cap \mathcal{K}$ is a partial subgroup of \mathcal{H} , and if $\mathcal{K} \trianglelefteq \mathcal{M}$ then $\mathcal{H} \cap \mathcal{K}$ is a partial normal subgroup of \mathcal{H} .

Proof. Let $w \in \mathbf{W}(\mathcal{H} \cap \mathcal{K}) \cap \mathbf{D}(\mathcal{H})$. Then $\Pi_{\mathcal{M}}(w) = \Pi_{\mathcal{H}}(w) \in \mathcal{H}$. As $\mathbf{D}(\mathcal{H}) \subseteq \mathbf{D}(\mathcal{M})$ and $\mathcal{K} \leq \mathcal{M}$ we have also $\Pi(w) \in \mathcal{K}$. Thus $\mathcal{H} \cap \mathcal{K} \leq \mathcal{H}$ and $\mathcal{H} \cap \mathcal{K} \preceq \mathcal{K}$. Transitivity of \preceq yields $\mathcal{H} \cap \mathcal{K} \preceq \mathcal{M}$. Now suppose that $\mathcal{K} \trianglelefteq \mathcal{M}$, and let $x \in \mathcal{H} \cap \mathcal{K}$ and $h \in \mathcal{H}$ such that $(h^{-1}, x, h) \in \mathbf{D}(\mathcal{H})$. Then $\Pi_{\mathcal{H}}(h^{-1}, x, h) \in \mathcal{H}$. But also $(h^{-1}, x, h) \in \mathbf{D}(\mathcal{M})$, so that $x^h \in \mathcal{K}$. Thus $\mathcal{H} \cap \mathcal{K} \trianglelefteq \mathcal{H}$. \square

Example 9.3. Let (\mathcal{H}, Γ) be a localizable pair in a locality (\mathcal{L}, Δ, S) . Then \mathcal{H}_Γ is an im-partial subgroup of \mathcal{L} .

For the remainder of this section (\mathcal{L}, Δ, S) will be a regular locality on \mathcal{F} . Denote by \mathcal{L}^s the expansion of \mathcal{L} to a proper locality whose set of objects is \mathcal{F}^s , and by \mathcal{L}^c the restriction of \mathcal{L}^s to a proper locality on \mathcal{F} whose set of objects is \mathcal{F}^c . Set

$$\mathcal{F}^*(\mathcal{F}) = \mathcal{F}_{S \cap F^*(\mathcal{L})}(F^*(\mathcal{L})) \quad \text{and} \quad E(\mathcal{F}) = \mathcal{F}_{S \cap E(\mathcal{L})}(E(\mathcal{L})).$$

For any proper locality \mathcal{L}' on \mathcal{F} that can be obtained from \mathcal{L} by a process of expansions and restrictions, define $F^*(\mathcal{L}')$ and $E(\mathcal{L}')$ by means of Theorem II.A2.

For any subgroup $X \leq S$ write \mathcal{F}_X for $N_{\mathcal{F}}(X)$. If X is fully normalized in \mathcal{F} then by II.6.3 there exists a proper locality $(\mathcal{L}_X^c, (\mathcal{F}_X)^c, N_S(X))$ on \mathcal{F}_X , and in fact the proof of II.6.3 involves the construction of \mathcal{L}_X^c as an im-partial subgroup of the locality \mathcal{L}^c . Write \mathcal{L}_X^s for the expansion of \mathcal{L}_X^c to a proper locality on \mathcal{F}_X whose set of objects is $(\mathcal{F}_X)^s$, and \mathcal{L}_X for the restriction of \mathcal{L}_X^s to a regular locality on \mathcal{F}_X .

Lemma 9.4. *Let X be fully normalized in \mathcal{F} , and let P be a subgroup of $N_S(X)$ containing X . Suppose that $P \in (\mathcal{F}_X)^s$. Then $P \in \mathcal{F}^s$.*

Proof. By II.6.3 \mathcal{F}_X is the fusion system of a proper locality. Subgroups of $N_S(P)$ which are fully normalized in \mathcal{F}_X are then fully centralized in \mathcal{F}_X , by II.1.13. As $P \in (\mathcal{F}_X)^s$, there exists an \mathcal{F}_X -conjugate Q of P such that Q is fully centralized in \mathcal{F}_X , and then Q is fully centralized in \mathcal{F} by II.1.16. Set $R = O_p(C_{\mathcal{F}}(Q))$. Then, since $Q \in (\mathcal{F}_X)^s$, II.6.4 yields $R \in C_{\mathcal{F}_X}(Q)^c$. As $C_{\mathcal{F}_X}(Q) = C_{\mathcal{F}}(Q)$, II.6.4 yields also $Q \in \mathcal{F}^s$, and thus $P \in \mathcal{F}^s$. \square

Corollary 9.5. *Let X be fully normalized in \mathcal{F} , and set $\Gamma = (\mathcal{F}_X)^s$. Then $(N_{\mathcal{L}^s}(X), \Gamma)$ is a localizable pair, and the locality $(N_{\mathcal{L}}(X)_\Gamma, \Gamma, N_S(X))$ given by 1.2 is isomorphic to \mathcal{L}_X^s .*

Proof. We note first of all that \mathcal{F}_X is the fusion system of a proper locality by 6.3, hence Γ is \mathcal{F}_X -closed by 6.7. Secondly, $N_S(X)$ is a maximal p -subgroup of $N_{\mathcal{L}}(S)$ as a straightforward consequence of I.2.11(b) and of the hypothesis that X is fully normalized in \mathcal{F} . Thirdly, for any $w \in \mathbf{W}(N_{\mathcal{L}}(X))$ with $N_{S_w}(X) \in \Gamma$ we have $N_{S_w}(X) \in \mathcal{F}^s$ by 9.4. This verifies the three conditions for a localizable pair in definition 1.1, and so 1.2 applies and yields a locality $(N_{\mathcal{L}}(X)_\Gamma, \Gamma, N_S(X))$. By 1.4(b) this locality is in fact a proper locality on \mathcal{F}_X . Further, $(N_{\mathcal{L}}(X)_\Gamma)$ is an expansion of the locality \mathcal{L}_X^c constructed in I.6.3, and is therefore isomorphic to \mathcal{L}_X^s by Theorem II.A1. \square

In what follows, we shall always identify \mathcal{L}_X^s with the locality $N_{\mathcal{L}^s}(X)_\Gamma$ in 9.5, and \mathcal{L}_X with the restriction of \mathcal{L}_X^s to $\delta(\mathcal{F}_X)$.

Corollary 9.6. *We have $\mathcal{L}_X \preccurlyeq \mathcal{L}_X^s \preccurlyeq \mathcal{L}^s$.*

Proof. Immediate from 9.3. \square

Lemma 9.7. *Let X be fully normalized in \mathcal{F} . Then $X \in \mathcal{F}^s$ if and only if $E(\mathcal{L}_X) = \mathbf{1}$.*

Proof. Set $P = O_p(\mathcal{L}_X)$, and note that $P = O_p(\mathcal{F}_X)$ by II.2.3. Suppose first that $E(\mathcal{L}_X) = \mathbf{1}$. Then $F^*(\mathcal{L}_X) = P$ by an application of 8.5 to the regular locality \mathcal{L}_X . Thus P is a large partial normal subgroup of \mathcal{L}_X , and so $C_{\mathcal{L}_X}(P) \leq P$. Then $P \in \mathcal{F}^c$ by II.6.2, and thus $X \in \mathcal{F}^s$. Conversely, suppose that $X \in \mathcal{F}^s$, so that $P \in \mathcal{F}^c$. Then $S \cap E(\mathcal{L}_X) = \mathbf{1}$, and then the regular locality $E(\mathcal{L}_X)$ is trivial. \square

Lemma 9.8. *Let \mathcal{H} be a product of components of \mathcal{L} , and let \mathcal{K} be the product of those components of \mathcal{L} which are not contained in \mathcal{H} . Assume that $\mathcal{H} \trianglelefteq \mathcal{L}$, and let D be a subgroup of $S \cap \mathcal{H}^\perp$ such that D is fully normalized in \mathcal{F} and such that $S \cap \mathcal{K} \leq D$. Then $S \cap \mathcal{H} \leq E(\mathcal{L}_D)$.*

Proof. Set $\Gamma = \{P \in \Delta \mid D \trianglelefteq P\}$, set $\mathcal{E} = \mathcal{F}_{S \cap \mathcal{H}}(\mathcal{H})$, and set $V = S \cap \mathcal{K}$. Notice that $E(\mathcal{L}) = \mathcal{H}\mathcal{K}$ and that $[\mathcal{H}, \mathcal{K}] = \mathbf{1}$, by 8.5. Then 2.10 applies to the regular locality $E(\mathcal{L})$, and 2.10(e) yields the following result.

(1) Let $U \leq S \cap \mathcal{H}$. Then $UV \in \Delta$ and $UD \in \Gamma$ if and only if $U \in \mathcal{E}^s$.

In particular, Γ is non-empty, and then 1.2 shows that $(N_{\mathcal{L}}(D), \Gamma)$ is a localizable pair. Write $\mathcal{L}_{(D, \Gamma)}$ for the locality $(N_{\mathcal{L}}(D)_\Gamma, \Gamma, N_S(D))$.

Let $w \in \mathbf{W}(\mathcal{H})$ and set $P = N_{S_w}(D)$. Then $D \trianglelefteq P$, and (1) shows that $P \in \Gamma$ if and only if $S_w \cap \mathcal{H} \in \mathcal{E}^s$. Thus $P \in \Gamma$ if and only if $w \in \mathbf{D}(\mathcal{H})$, and this shows that \mathcal{H} is a partial subgroup of $\mathcal{L}_{(D, \Gamma)}$. As $\mathcal{H} \trianglelefteq \mathcal{L}$ we obtain:

(2) $\mathcal{H} \trianglelefteq \mathcal{L}_{(D, \Gamma)}$.

As $\mathcal{H} \trianglelefteq \mathcal{L}$ we have $S \cap \mathcal{H}$ strongly closed in \mathcal{F} , and hence $S \cap \mathcal{H}$ is strongly closed in \mathcal{F}_D . The hypothesis of II.6.10 is then fulfilled, with the subcentric locality $(\mathcal{L}_D^s, (\mathcal{F}_D)^s, N_S(D))$ on \mathcal{F}_D in the role of \mathcal{L} and with $S \cap \mathcal{H}$ in the role of T . We may therefore conclude that for each $P \in (\mathcal{F}_D)^{cr}$ there exists an \mathcal{F}_D -conjugate Q of P with $Q \cap \mathcal{H} \in \mathcal{E}^c$. Then $Q \in \Gamma$ by (1), and then $(\mathcal{F}_D)^{cr} \subseteq \Gamma$ as Γ is \mathcal{F}_D -closed. Then 1.4(b) yields:

(3) $\mathcal{L}_{(D, \Gamma)}$ is a proper locality on \mathcal{F}_D .

We may now apply Theorem II.A to $\mathcal{L}_{(D, \Gamma)}$. Let \mathcal{M}^+ be a large partial normal subgroup of \mathcal{L}_D^s containing $O_p(\mathcal{F}_D)$, and set $\mathcal{M} = \mathcal{M}^+ \cap \mathcal{L}_{(D, \Gamma)}$. Then $S \cap \mathcal{M} = S \cap \mathcal{M}^+$, and $\mathcal{H} \cap \mathcal{M} \trianglelefteq \mathcal{L}_{(D, \Gamma)}$. Set $\mathcal{H}_0 = C_{\mathcal{H}}(\mathcal{H} \cap \mathcal{M})$. As $\mathcal{H} \cap \mathcal{M} \trianglelefteq \mathcal{H}$ where \mathcal{H} is a regular locality, 7.7(c) yields $\mathcal{H}_0 \trianglelefteq \mathcal{H}$. Then $\mathcal{H}_0 = Z(\mathcal{H})\mathcal{H}_1$ where $\mathcal{H}_1 = O^p(\mathcal{H}_1)$ is a product of components of \mathcal{H} , by 8.6. As $\mathcal{L}_{(D, \Gamma)}$ is a subset of $N_{\mathcal{L}}(S \cap \mathcal{H})$, it follows from 7.7(e) that $\mathcal{L}_{(D, \Gamma)}$ acts on \mathcal{H} by conjugation. This action plainly preserves \mathcal{H}_0 and \mathcal{H}_1 . Thus $\mathcal{H}_1 \trianglelefteq \mathcal{L}_{(D, \Gamma)}$. We now claim (see 3.7):

(4) $\Sigma_{S \cap \mathcal{H}_1}(\mathcal{F}_D) \subseteq \Gamma$.

For this it suffices to show that $DXY \in \Gamma$ for $X \in \mathcal{F}_{S \cap \mathcal{H}_1}(\mathcal{H}_1)^{cr}$ and $Y \in C_{\mathcal{F}_D}(S \cap \mathcal{H}_1)^{cr}$. For any such X and Y we have $XY \in \mathcal{E}^{cr}$ and $VXY \in E(\mathcal{F})^{cr}$ by 2.10(d), so $VXY \in \Delta$. Thus $DXY \in \Gamma$, and (4) holds.

We have $[\mathcal{H}_1, S \cap \mathcal{M}] \leq \mathcal{H}_1 \cap \mathcal{M} \leq Z(\mathcal{H}_1)$, and then $[\mathcal{H}_1, S \cap \mathcal{M}] = \mathbf{1}$ since $\mathcal{H}_1 = O^p(\mathcal{H}_1)$. Let \mathcal{H}_1^+ be the partial normal subgroup of \mathcal{L}_D^s whose intersection with $\mathcal{L}_{(D, \Gamma)}$ is \mathcal{H}_1 . Then \mathcal{H}_1^+ is the partial subgroup $\langle \mathcal{H}_1 \rangle$ of \mathcal{L}_D^s generated by \mathcal{H}_1 , by (4) and 3.14(b),

and it follows that $[\mathcal{H}_1^+, S \cap \mathcal{M}] = \mathbf{1}$. As $\mathcal{H}_1^+ = O^p(\mathcal{H}_1^+)$ by II.7.3, it follows from definition 5.5 that $\mathcal{H}_1^+ \leq (\mathcal{M}^s)^\perp$ (where the “ \perp ” operation is taken in the locality \mathcal{L}_D^s). As \mathcal{M}^s is large in \mathcal{L}_D^s we have $S \cap (\mathcal{M}^s)^\perp = Z(\mathcal{M}^s)$ by 6.4, and thus $S \cap \mathcal{H}_1 \leq \mathcal{M}$. As $\mathcal{H}_1 \cap \mathcal{M} \leq Z(\mathcal{H}_1)$ we conclude that $\mathcal{H}_1 = \mathbf{1}$ and that $\mathcal{H} \leq \mathcal{M}$. Thus $S \cap \mathcal{H} \leq F^*(\mathcal{L}_D^s)$. As $S \cap F^*(\mathcal{L}_D^s) = S \cap F^*(\mathcal{L}_D)$ by Theorem II.A, the proof is complete. \square

We are now in position to prove a version for regular localities of the “ L -balance” Theorem from finite group theory. For a discussion of this result for finite groups see [GLS], and for a fusion-theoretic version see [Asch].

Theorem 9.9. *Let \mathcal{L} be a regular locality $X \leq S$ be fully normalized in \mathcal{F} . Then $S \cap E(\mathcal{L}_X) \leq E(\mathcal{L})$. Moreover, if $X \leq F^*(\mathcal{L})$ then $\mathcal{L}_X \preccurlyeq \mathcal{L}$ and $E(\mathcal{L}_X) \preccurlyeq E(\mathcal{L})$.*

Proof. Assume false, and let X be a counterexample of maximal order. set $R = S \cap E(\mathcal{L})$, and let $(\mathcal{K}_1, \dots, \mathcal{K}_m)$ be a listing (in arbitrary order) of the components of \mathcal{L}_X . As $[\mathcal{K}_i, \mathcal{K}_j] = \mathbf{1}$ for all i and j with $i \neq j$, I.5.1 shows that $S \cap E(\mathcal{L}_X)$ is the product in any order of the intersections $S \cap \mathcal{K}_i$. Let \mathcal{H} be the product of all of the components \mathcal{K}_i such that $S \cap \mathcal{K}_i \not\leq E(\mathcal{L})$. As $\mathcal{L}_X \preccurlyeq \mathcal{L}^s$ we have $\mathcal{H} \cap E(\mathcal{L}^s) \leq \mathcal{H}$ by 9.2(b), and then $\mathcal{H} \cap E(\mathcal{L}^s) \leq Z(\mathcal{H})$ by 8.6.

Set $A = N_R(X)$. Then $[S \cap \mathcal{H}, A] \leq Z(\mathcal{H})$, so A normalizes every P in $\mathcal{F}_{S \cap \mathcal{H}}(\mathcal{H})^{cr}$. For each such P , 9.2(b) yields

$$[N_{\mathcal{H}}(P), A] \leq N_{\mathcal{H}}(P) \cap E(\mathcal{L}) \leq Z(\mathcal{H}).$$

Thus $N_{\mathcal{H}}(P)$ centralizes the chain $Z(H)A \geq Z(A) \geq \mathbf{1}$, and then $[O^p(N_{\mathcal{H}}(P)), A] = \mathbf{1}$ by coprime action (II.2.5). By 5.2 $O^p(\mathcal{H})$ is generated by the set of all $O^p(N_{\mathcal{H}}(P))$ with P as above, and then $[\mathcal{H}, A] = \mathbf{1}$ as $\mathcal{H} = O^p(\mathcal{H})$. Thus $[\mathcal{H}, AX] = \mathbf{1}$.

Let $D \in (AX)^{\mathcal{F}}$ be fully normalized in \mathcal{F} . As \mathcal{F} is inductive there exists an \mathcal{F} -homomorphism $\phi : N_S(AX) \rightarrow N_S(D)$ such that $(AX)\phi = D$. Set $Y = X\phi$ and $B = A\phi$. Then $B \leq N_R(Y)$, and equality then holds since X is fully normalized in \mathcal{F} , and since R is strongly closed in \mathcal{F} . As $N_S(X) \leq N_S(AX)$, ϕ maps $N_S(X)$ into $N_S(Y)$, and so Y is fully normalized in \mathcal{F} . We observe that the \mathcal{F} -isomorphism $N_S(X) \rightarrow N_S(Y)$ induced by ϕ induces also an isomorphism $\mathcal{F}_X \rightarrow \mathcal{F}_Y$ of fusion systems, by II.1.5. Since $S \cap E(\mathcal{L}_X)$ and $S \cap E(\mathcal{L}_Y)$ depend only on \mathcal{F}_X and \mathcal{F}_Y , we may now assume that $X = Y$ and that D is fully normalized in \mathcal{F} . Moreover, from the preceding paragraph we have:

$$(1) \quad \mathcal{H} \leq C_{E(\mathcal{L}_X)}(D).$$

Note that $R \in \mathcal{F}^s$ by II.6.9. If $R \leq X$ then also $X \in \mathcal{F}^s$, and then 9.7 yields a contradiction to the non-triviality of \mathcal{H} . Thus $R \not\leq X$, and so X is a proper subgroup of D . The maximality of $|X|$ then implies that D is not a counterexample to the lemma, and so:

$$(2) \quad S \cap E(\mathcal{L}_D) \leq E(\mathcal{L}).$$

Set $S_{D,X} = N_S(D) \cap N_S(X)$, and let $\mathcal{F}_{D,X}$ be the fusion system $N_{N_{\mathcal{F}}(D)}(X)$ on $S_{D,X}$. It is a straightforward consequence of the definition (following II.1.4) of normalizers in fusion systems that $\mathcal{F}_{D,X} = N_{N_{\mathcal{F}}(X)}(D)$. Set $\Gamma_{D,X} = (\mathcal{F}_{D,X})^s$, and write

$(\mathcal{L}_{D,X}^s, \Gamma_{D,X}, S_{D,X})$ for the proper locality on $\mathcal{F}_{D,X}$ given by two applications of 9.5. By 9.4, a word $w \in \mathbf{W}(N_{\mathcal{L}}(D) \cap N_{\mathcal{L}}(X))$ is in $\mathbf{D}(\mathcal{L}_{D,X}^s)$ if and only if $S_w \cap S_{D,X} \in \mathcal{F}^s$. Thus:

$$(3) \quad \mathcal{L}_{D,X}^s \preceq \mathcal{L}_D^s \preceq \mathcal{L}^s.$$

We may now appeal to lemma 9.8 with \mathcal{L}_X in the role of \mathcal{L} , in order to obtain $S \cap \mathcal{H} \leq E(\mathcal{L}_{D,X}^s)$. We may write also:

$$(4) \quad S \cap \mathcal{H} \leq E(\mathcal{L}_{D,X}),$$

where $\mathcal{L}_{D,X}$ is the regular locality obtained by restriction from $\mathcal{L}_{D,X}^s$.

Suppose that $D \leq O_p(\mathcal{L})$. Then $X \leq O_p(\mathcal{L})$, and so $X \leq E(\mathcal{L})^\perp$. This yields $A = R$, so that $R \leq D \leq O_p(\mathcal{L})$. Then $E(\mathcal{L}) = \mathbf{1}$, and $F^*(\mathcal{L}) = O_p(\mathcal{L})$. Thus $O_p(\mathcal{L}) \in \Delta$, and \mathcal{L} is a group of characteristic p . Then $\mathcal{L}_X = N_{\mathcal{L}}(X)$ is a group of characteristic p by II.2.7(b), and $E(\mathcal{L}_X) = \mathbf{1}$. As there is nothing to prove in this case, we may assume that $D \not\leq O_p(\mathcal{L})$.

By induction on $|\mathcal{L}|$ we may then assume that the Theorem holds for \mathcal{L}_D in the role of \mathcal{L} . Thus:

$$(5) \quad S \cap E(\mathcal{L}_{D,X}) \leq S \cap E(\mathcal{L}_D).$$

Suppose next that $X = D$. Then $R \leq X$, so $O_p(\mathcal{L})X \in \Delta \subseteq \mathcal{F}^s$, and then $X \in \mathcal{F}^s$ by II.6.9. Then $E(\mathcal{L}_X) = \mathbf{1}$ by 9.7, and there is again nothing to prove. Thus we may assume that X is a proper subgroup of D , and the maximal choice of X then yields $S \cap E(\mathcal{L}_D) \leq E(\mathcal{L})$. This result, in combination with (4) and (5), now yields $S \cap \mathcal{H} \leq E(\mathcal{L})$. By definition, \mathcal{H} is the product of those components \mathcal{K} of \mathcal{L} such that $S \cap \mathcal{K} \not\leq E(\mathcal{L})$, we conclude that $\mathcal{H} = \mathbf{1}$, and that $S \cap E(\mathcal{L}_X) \leq E(\mathcal{L})$. This completes the first part of the proof.

Suppose now that $X \leq F^*(\mathcal{L})$. Let $w \in \mathbf{D}(\mathcal{L}_X)$, and set $P = N_{S_w}(X) \cap F^*(\mathcal{L})$. Then $X \leq P$, and $S_w \cap E(\mathcal{L}_X) \leq P$ by what has just been proved. Here $S_w \cap F^*(\mathcal{L}_X) \in \mathcal{F}_X^s$ by the definition of $\mathbf{D}(\mathcal{L}_X)$, and then $S_w \cap E(\mathcal{L}_X) \in \mathcal{F}_X^s$ by II.6.9. Thus $X \leq P \in \mathcal{F}_X^s$, and then $P \in \mathcal{F}^s$ by 9.4. Thus $P \in \delta(\mathcal{F})$. Recall that $\mathcal{L}_X \preceq \mathcal{L}^s$ by definition. In the case that $w = (g)$ for some $g \in \mathcal{L}_X$ we now conclude that $g \in \mathcal{L}$. Thus $\mathcal{L}_X \subseteq \mathcal{L}$, and $\mathbf{D}(\mathcal{L}_X) \subseteq \mathbf{D}(\mathcal{L})$. This yields $\mathcal{L}_X \preceq \mathcal{L}$.

We have already seen that $S \cap E(\mathcal{L}_X) \leq E(\mathcal{L})$. As $E(\mathcal{L}_X) \preceq \mathcal{L}$ we have $E(\mathcal{L}_X) \cap E(\mathcal{L}) \trianglelefteq E(\mathcal{L}_X)$ by 9.2(b). Each component \mathcal{K} of \mathcal{L}_X has the property that $\mathcal{K}/Z(\mathcal{K})$ is simple, so \mathcal{K} has no proper partial normal subgroups containing $S \cap \mathcal{K}$. This shows that $E(\mathcal{L}_X) \subseteq E(\mathcal{L})$, and then $E(\mathcal{L}_X) \preceq E(\mathcal{L})$ by 9.2(a). \square

Section 10: Theorems C, D, and E

Theorem C is given by Theorems 5.7 and 7.7 (which in fact contain a great deal more information), and Theorem D is Theorem 8.5.

Let (\mathcal{L}, Δ, S) be a regular locality on \mathcal{F} . Point (a) of Theorem E is that if $\mathcal{N} \trianglelefteq \mathcal{L}$ is a partial normal subgroup of \mathcal{L} then \mathcal{N} is a regular locality; which is already part of Theorem C. Point (b) of Theorem E, concerning the regular locality \mathcal{L}_X on $N_{\mathcal{F}}(X)$,

under the assumption that X is fully normalized in \mathcal{F} and that $X \leq F^*(\mathcal{L})$, is given by Theorem 9.9. Point (c) of Theorem E is Proposition 7.2.

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